Helen A. Merrill
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SCIENCE AND ART DRAWING

COMPLETE GEOMETRICAL COURSE

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CONSISTING OF

PLANE AND SOLID GEOMETRY, ORTHOGRAPHIC AND ISOMETRIC PROJECTION, PROJECTION OF SHADOWS, THE PRINCIPLES OF MAP PROJECTION, GRAPHIC ARITHMETIC, AND GRAPHIC STATICS

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PREFACE

Practical Geometry is the weapon which they must use who would attack practical work. The Engineer, the Architect, the Soldier, and the Statistician, all have recourse to its assistance to solve their problems or to explain their methods. Every day the graphic treatment of subjects is finding its application in new directions; and to be able to delineate the proportions of any subject places in the hand an invaluable tool for the execution of any design of practical value.

The Author, in adding to the many excellent manuals on the subject which have been already published one more of yet untried merit, appeals to those particularly who wish to have a solid grounding without the aid of a teacher, and also to those who need assistance in passing from the simpler to the more intricate parts of the subject. His publishers have spared no expense in enabling him to elaborate those parts which are known to present real difficulties to the student. There has been no demand upon him to abridge the necessary information, the scantiness of which so often makes it difficult for the student to comprehend the more advanced portions of the subject.

A free use has been made of perspective sketches, which are good substitutes for models; and in several instances a practical method of obtaining the results has been introduced in addition to that obtained by geometrical construction; *e.g.* all the conic

sections could be obtained with the greatest accuracy by the method illustrated at the commencement of Chapter XV.

In practice subjects are sometimes treated in a conventional manner; for instance, illustrated in the first part of Chapter XXXIII. is the conventional method of projecting shadows in architectural and engineering drawings.

These additions have been made to make the book useful, not only to students, but also to those engaged in practical work.

This book is intended to embrace the whole course of practical geometry required in various examinations, and it includes the *Principles of Map Projection, Graphic Arithmetic,* and *Graphic Statics.*

The Syllabus of the Geometrical Course for the South Kensington Science and Art Examinations is given in detail, with references to the problems contained in this work (page 575).

This course covers nearly the whole ground necessary for the following examinations:—

The Royal College of Science, and School of Mines.
The City of London College.
The College of Preceptors.
The Army Examinations.
The Oxford and Cambridge Local Examinations.
The Indian Engineering College, Cooper's Hill.

Where any of these examinations contain certain special portions which are not named in the South Kensington Syllabus, these can be found by reference to the index of this book, *e.g.* the use of sector (Cooper's Hill Syllabus), page 132.

The Author begs to acknowledge his indebtedness to the

following text-books, which have furnished many invaluable hints towards the execution of his design, viz. Angel's *Practical Plane Geometry and Projection*, Bradley's *Elements of Practical Geometry*, Pressland's *Geometrical Drawing*, and Jessop's *Elements of Applied Mechanics.*

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PRACTICAL PLANE GEOMETRY

DRAWING INSTRUMENTS AND MATERIALS.

The Drawing-board.—A very convenient size to use for ordinary purposes is half Imperial ($23'' \times 16''$); it should be made of well-seasoned yellow pine, with the edges true and at right angles to each other.

The Tee-square.—This is a ruler with a cross piece or stock at the end: it is like the letter T in shape, hence its name. The blade should be screwed on to the stock, and not mortised into it, so as to allow of the set-squares being used up to the extreme margin of the paper, as illustrated under the head of *General Directions* (p. 4). By keeping the stock of the tee-square pressed closely against the edge of the drawing-board, we are enabled to draw lines parallel to each other.

Set-squares.—These are right-angled triangles made with given fixed angles out of thin pieces of wood or ebonite: the latter is preferable, as it is not liable to warp. The most useful angles are those of $45°$ and $60°$.

French Curves.—These are thin pieces of wood cut into a variety of curves. They are used for drawing fair curves, that are not arcs of circles, through a succession of points: the cycloidal curves, for instance.

Scale.—A plain scale about 6 inches long, divided into inches, with sub-divisions of eighths on one edge and tenths on the other.

Pencils.—Two degrees of hardness should be used: HH for drawing in the construction, and F for drawing in the result with a firmer line.

Drawing-paper.—This should have a hard smooth surface. Whatman's "*hot-pressed*" is the best for fine work; but if the drawing has to be coloured, a damp sponge should first be drawn across the surface, to remove the gloss. Cartridge-paper of good quality is suitable for ordinary work.

The most convenient size is "*Imperial*" (30″ × 22″), which can be cut to half, or quarter Imperial, as desired.

Drawing-pins.—These should have short fine points, so as not to make large holes in the drawing-board.

Dividers.—These are also called compasses, and are used for setting off distances or dividing lines. There is a special kind made, called "hair-dividers," one leg of which can be adjusted by means of a spring and screw: these are very useful for dividing lines, etc.

Bow-pencil.—This is a small pair of compasses with one leg constructed to hold a pencil: it is used for drawing circles and arcs.

Bow-pen.—This is a similar instrument to a bow-pencil, but has a ruling pen for one of its legs instead of a pencil, and is used for inking in circles and arcs.

Note.—Both the bow-pencil and bow-pen should have hinged legs; because, when a number of circles are drawn from the same centre, they are likely to make a large hole in the paper, unless the leg used for the centre is kept perpendicular to the paper. It is also necessary to have the pen-leg as upright as possible, otherwise it has a tendency to draw uneven lines.

Ruling-pen.—This is used for inking in lines, the thickness of which is regulated by a screw. Some are made in which the nib that works against the ruler is of an extra thickness of metal: this is to prevent the nibs from closing when the pen is pressed against the ruler.

Indian ink should be used for inking in a drawing. It has several advantages over common ink: it dries quickly; it does not corrode the ruling-pen; and the lines can be coloured over without their running.

The most convenient is the liquid Indian ink, sold in bottles, as it is always ready for use. The ruling-pen should be filled with Indian ink by means of an ordinary steel nib. If the cake Indian ink is used, after rubbing it in a saucer, a piece of thin whalebone should be used for filling the ruling-pen.

GENERAL DIRECTIONS.

Keep all instruments perfectly clean: do not leave ink to dry in the ruling-pen.

In using dividers avoid, as much as possible, making holes through the paper.

The paper should be firmly fixed to the drawing-board by a drawing-pin at each corner, well pressed down. Do not stick pins in the middle of the board, because the points of the dividers are liable to slip into them and make unsightly holes in the paper.

A pencil sharpened to what is called a "chisel-point" is generally used for drawing lines; it has the advantage of retaining its point longer, but a nicely-pointed pencil is better for neat work, as it enables us to see the commencement and termination of a line more easily.

Always rule a line from *left to right*, and slope the pencil slightly towards the direction in which it is moving; if this is done, there is less chance of indenting the paper, which should always be avoided.

Having determined the extent of a line, always rub out the superfluous length; this will prevent unnecessary complication.

Avoid using India-rubber more than is necessary, as it tends to injure the surface of the paper. After inking in a

drawing, use stale bread in preference to India-rubber for cleaning it up.

The tee-square should be used for drawing horizontal lines only; the perpendicular lines should be drawn by the set-

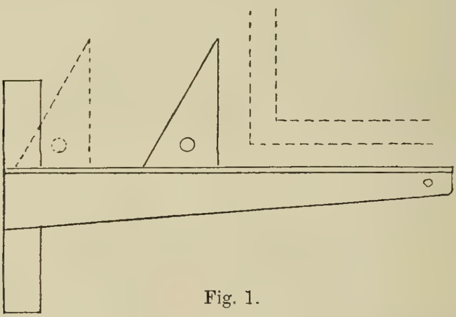


Fig. 1.

squares. If this is done, it is immaterial whether the edges of the drawing-board are right angles, because it will only be necessary to use one of its edges.

For drawing parallel lines that are neither horizontal nor perpendicular, hold one set-square firmly pressed upon the

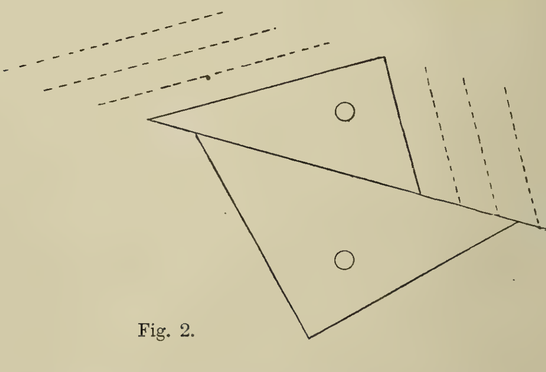


Fig. 2.

paper and slide the other along its edge. Geometrical drawing can be greatly facilitated by the proper use of set-squares, so it is advisable to practise their use.

When a problem contains many arcs of circles, it is advisable to connect the arc with its corresponding centre. Enclose the centre in a small circle; draw a dotted line to the arc, terminated by an arrow-head (Fig. 3).

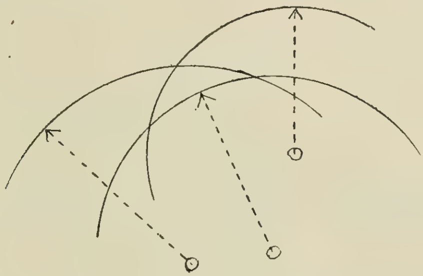


Fig. 3.

In drawing intersecting arcs for bisecting lines, etc., the arcs should not intersect each other too obtusely or too acutely: the nearer the angle between the arcs approaches $90°$ the easier it will be to ascertain the exact point required.

In joining two points by a line, first place the point of the pencil on one point, then place the edge of the ruler against it, and adjust the ruler till its edge coincides with the other point.

All the problems should be drawn larger than shown. Where possible, it is advisable to vary the conditions,—for instance, in *Principles of Similitude,* select figures of a different shape to those shown. In the *General Methods* for drawing the regular polygons, apply the method to drawing several of the polygons: do not be satisfied with one example.

Great accuracy is required in drawing the various problems. Every effort should be made to ensure neatness and precision in the work.

All arcs should be inked in first, as it is easier to join a line to an arc than an arc to a line.

CHAPTER I

DEFINITIONS

A *point* simply marks position—it is supposed to have no magnitude.

A *line* has length only, and no thickness: the extremities and intersection of lines are points. A *straight line* is sometimes called a *right line,* and is the shortest distance between two points.

To *produce* a line is to lengthen it.

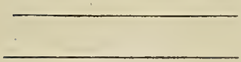


Fig. 4.

Parallel lines are an equal distance apart throughout their entire length, and if produced in either direction would never meet.

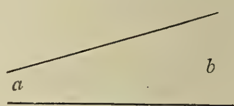


Fig. 5.

Lines drawn thus, are said to *converge* towards *a* and *diverge* towards *b*. If we were to produce the ends at *a* till they meet, they would form an *angle.*

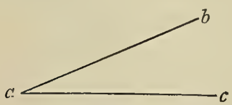


Fig. 6.

An *angle* is the amount of opening between two lines which meet at one point. As the point *a* is common to both lines, the angle here shown would be called *bac.*

An angle is measured by degrees, which are expressed by a small circle over the figures—thus 22° means 22 degrees.

Lines are said to be *bisected* when divided into two equal parts.

A *vertical line* at its lower end points towards the centre

of the earth, in the same direction as a string suspended with a weight attached to it.

A *horizontal line* is a line parallel to the surface of the earth, *i.e.* at right angles to a vertical line.

A *circle* is a curved line drawn round a common point or centre, and every part of it is an equal distance from this point. This curved line is called the "*circumference*" or "*periphery*" of the circle. A line drawn through the centre till it meets the circumference on each side, as *ab*, is called the "*diameter*"; and any line drawn from the centre to the circumference, as *ce* or *cd*, is called a "*radius*." A circle contains 360°, so the angle *acd* would contain 90°, or one-quarter of the whole. This angle is called a "*right angle*," shown by the dotted curve. The angle *ace* contains 45°, or one-eighth of the circle, while *ecb* contains 135°.

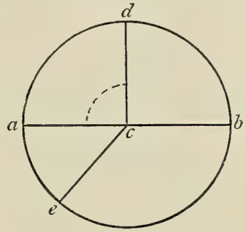


Fig. 7.

All angles containing more than 90° are called "*obtuse angles*," as *ecb*; while those containing less than 90° are called "*acute angles*," as *ace*.

The line *dc* is said to be *perpendicular*, or at right angles to *ab*, and is the distance of the point *d* from the line *ab*.

A *semicircle* is half a circle, as *adb*.

A *quadrant* is a quarter of a circle, as *ad*.

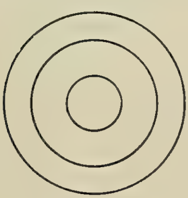


Fig. 8.

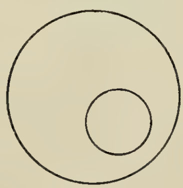


Fig. 9.

Concentric circles have the same centre, and are therefore parallel to each other (Fig. 8).

Eccentric circles have not the same centre (Fig. 9).

A *plane* is a flat surface which has length and breadth, but no thickness.

Area is the amount of surface a plane contains.

TRIANGLES.

Triangles are figures that have three angles, and are contained by three sides; they are named after certain specialities that they possess.

There are three kinds of triangles named after their sides: viz. *equilateral*, *isosceles*, and *scalene.*

An *equilateral triangle* has three equal sides (Fig. 10).

An *isosceles triangle* has two sides equal (Fig. 11).

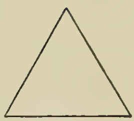


Fig. 10.

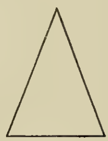


Fig. 11.

Fig. 12.

A *scalene triangle* has none of its sides equal (Fig. 12).

There are three kinds of triangles named after their angles: viz. a *right-angled* triangle, an *obtuse-angled* triangle, and an *acute-angled* triangle.

A *right-angled triangle* has one right angle (Fig. 13).

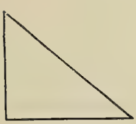


Fig. 13.

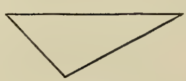


Fig. 14.

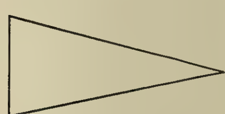


Fig. 15.

An *obtuse-angled triangle* has one obtuse angle (Fig. 14).

An *acute-angled triangle* has three acute angles (Fig. 15).

Parts of a Triangle.

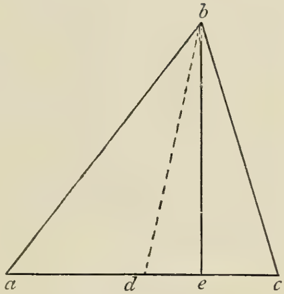


Fig. 16.

The *base* is its lowest side, as *ac*.

The *vertex* is the point opposite its base, as *b*.

The *altitude* or *perpendicular height* is a line drawn from the vertex, perpendicular to its base, as *be*.

The *median* is a line drawn from the vertex to the middle point of the base, as the dotted line *bd*.

The *hypotenuse* is the line opposite the right angle of a right-angled triangle.

Quadrilateral Figures.

Quadrilateral figures have four sides and are of the following kinds:—

A *square* has four equal sides and four equal angles (Fig. 17).


Fig. 17.


Fig. 18.

A *rectangle* or *oblong* has only its opposite sides equal, but has four equal angles (Fig. 18).

A *rhombus* has all its sides equal, but its angles are not right angles (Fig. 19).

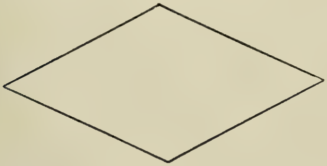


Fig. 19.

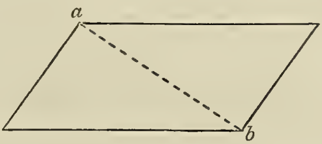


Fig. 20.

A *rhomboid* has only its opposite sides equal, and its angles are not right angles (Fig. 20).

The four figures just described are also called "*parallelograms*," their opposite sides being always parallel to each other, and of equal length; their opposite angles are also equal to each other. A line joining the opposite angles will always divide the figure into two equal parts: this line is called a "*diagonal*," as the dotted line *ab*.

If a parallelogram has one right angle, all its angles must be right angles.

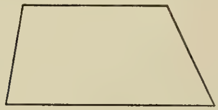


Fig. 21.

Fig. 22.

A *trapezoid* has only two of its sides parallel to each other (Fig. 21).

A *trapezium* has none of its sides parallel (Fig. 22).

Fig. 23.

Fig. 24.

A *right-angled trapezoid* has two right angles (Fig. 23).

An *isosceles trapezoid* has two opposite sides equal (Fig. 24).

Parts of a Circle.

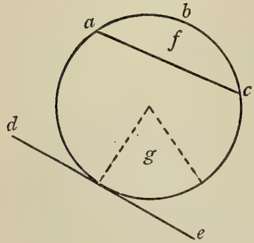


Fig. 25.

An *arc* is any portion of the circumference of a circle, as *abc*.

A *chord* is a straight line joining the extremities of an arc, as *ac*.

A *segment* is the space contained between the arc and the chord, as *f*.

A *sector* is the space enclosed by two radii and the arc, as *g*.

A *tangent* is a line touching the circumference in one point: it is always at right angles to the radius of the circle at that point, as *de*.

Ordinate is a line drawn from a point in a curve perpendicular to the diameter, as dotted line in Fig. 26 (1).

Abscissa is the part of the diameter cut off by the ordinate, as dotted line in Fig. 26 (2).

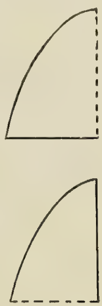


Fig. 26.

Polygons.

Polygons are figures that contain more than four sides, and are of two kinds, viz. *regular* and *irregular*.

Regular polygons have their sides of equal length and their angles also equal.

Irregular polygons have their sides as well as their angles unequal.

Pentagon	has	5	equal sides and	5	equal angles.
Hexagon	,,	6	,,	6	,,
Heptagon	,,	7	,,	7	,,
Octagon	,,	8	,,	8	,,
Nonagon	,,	9	,,	9	,,
Decagon	,,	10	,,	10	,,
Undecagon	,,	11	,,	11	,,
Duodecagon	,,	12	,,	12	,,

To find the angles of a regular polygon we divide 360° by the number of sides it contains; this will give the angles at the centre. Then by subtracting one of these angles from 180° the remainder will give the angle between its sides. For example, to find the angles of an octagon, divide 360° by $8 = 45°$, the angle at centre; 180° less $45° = 135°$, the angle formed by the sides of the octagon.

All the foregoing definitions refer to plane surfaces or parts of same. For instance, lines are parts of planes: a line drawn upon a plane will form part of it, and the intersection

of two planes will form a line. In the same way points are parts of lines, and the intersection of lines will give points.

SOLIDS.

These do not form part of Plane Geometry, but as they will be required in other sections of this work, it is advisable to include them in these definitions.

A *solid* has length, breadth, and thickness.

Solids are of great variety; they are also of regular and irregular shape.

Regular Solids have equal faces and equal edges, and cannot have fewer than four sides.

Tetrahedron is composed of four equilateral triangles.

Hexahedron or *cube* is formed of six equal squares.

Octahedron is contained by eight equilateral triangles.

Dodecahedron has twelve equal pentagons for its faces.

Icosahedron is composed of twenty equilateral triangles.

Prisms are solids whose ends are similar, the sides of which are parallelograms, and their edges parallel to each other; they are named after the shape of their bases.

Triangular prism, having a base of three sides.

Quadrangular prism, having a base of four sides.

Hexagonal prism, having a base of six sides.

etc.

Pyramids are solid figures, the edges of which meet at a vertex; they are also named after the shape of their bases.

Triangular pyramid.—The tetrahedron is a triangular pyramid, having three sides to its base.

Quadrangular pyramid, having four sides to its base.

Hexagonal pyramid, having six sides to its base.

etc.

Solids formed of Plane and Curved Surfaces.

A *cylinder* is a surface, every point of which is equally distant from a straight line called its *axis*.

A *cone* is a surface described by the revolution of a right-angled triangle about one of its sides called its *axis*.

Note.—A cylinder is a circular prism.
A cone is a circular pyramid.

Solids formed of Curved Surfaces only.

Sphere.—Every part of the surface is equally distant from its centre, and every section[1] of it is a circle.

Spheroid.—Resembling the sphere in shape, but all its sections are not circles.

GENERAL PROPERTIES

OF SOME OF THE FIGURES ALREADY DESCRIBED.

If two lines cross each other the opposite angles are always equal. The angle *acb* is equal to the angle *dce*, and the angle *acd* is equal to the angle *bce*.

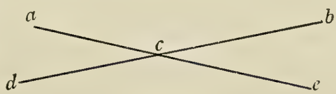


Fig. 27.

The two adjacent angles are equal to two right angles; the angles *acd* and *acb*, for instance, as well as the angles *bce* and *ecd*.

Triangles.

The three angles of a triangle contain together 180°, or two right angles; so if two angles are given, the third angle can always be found. For example, if one angle is 70° and the other 30°, the remaining angle must be 80°.

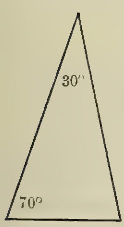


Fig. 28.

$$70° + 30° = 100°.$$
$$180° - 100° = 80°.$$

The exterior angle of a triangle is equal to the two opposite interior angles. The angle *abc* is

[1] A *section* literally means a part separated from the rest, but in practical geometry and the other subjects treated of in this work it really means *intersection:* a solid is supposed to be cut through by a plane and a part removed.

equal to the angles *bcd* and *cdb* together; in the same way the angle *cde* is equal to the two angles *bcd* and *cbd*.

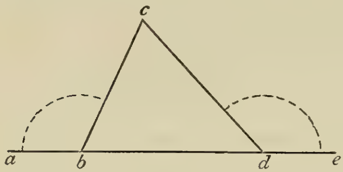


Fig. 29.

If we multiply the base by half the altitude, we get the *area* of a triangle; or half its base by its altitude will give us the same result.

Triangles of equal bases drawn between parallel lines are

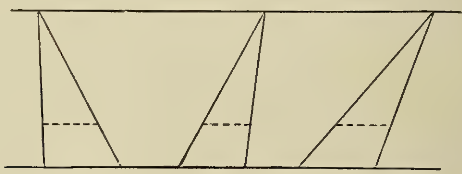


Fig. 30.

equal in area, and lines drawn parallel to their bases at equal heights are equal in length, as the dotted lines shown (Fig. 30).

If we bisect two sides of a triangle and join the points of bisection, we get a line that is always parallel to the third side (Fig. 31).

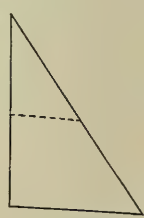


Fig. 31.

Quadrilaterals.

If we bisect the four sides of a quadrilateral figure and join the points, it will always give us a parallelogram, as shown by dotted lines. The reason for this will be apparent by the principle shown in the preceding figure if we draw a diagonal so as to form two triangles.

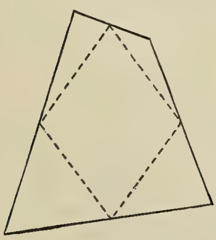


Fig. 32.

Parallelograms drawn between parallel lines on equal bases are always equal

in area, and parallel lines drawn at equal heights are always

Fig. 33.

equal to each other and to the bases, as shown by dotted lines.

Semicircles.

Any two lines drawn from the extremities of the diameter, to a point on the circumference of a semicircle, will form a right angle.

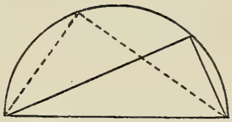


Fig. 34.

Tangents.

Tangents drawn to any three circles of different diameters all meet in the same straight line.

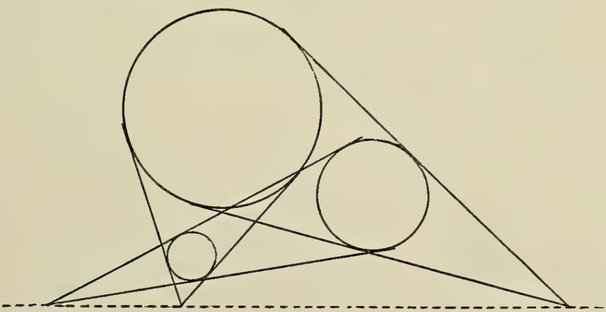


Fig. 35.

CHAPTER II

LINES, TRIANGLES, QUADRILATERALS, CONVERGENT LINES, AND CIRCLES

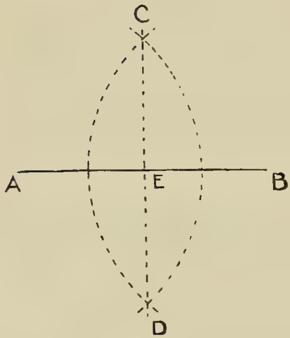


Fig. 36.

Problem 1.

To bisect a given straight line AB.

From A and B as centres, with any radius greater than half the line, describe arcs cutting each other in C and D; join CD by a line. This line will bisect AB in E. The line CD will be perpendicular to AB.

Problem 2.

To bisect a given arc AB.

Proceed in the same way as in Problem 1, using the extremities of the arc as centres. The arc AB is bisected at E.

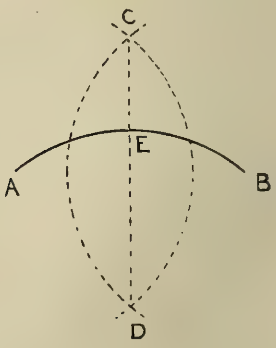


Fig. 37.

Problem 3.

To draw a line parallel to a given line AB *through a given point* C.

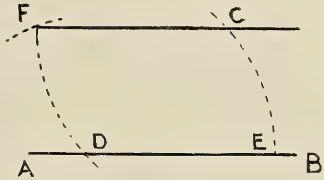


Fig. 38.

Take any point D in line AB, not opposite the point C; with D as centre, and DC as radius, describe an arc cutting AB in E, and from C as centre, with the same radius, draw another arc DF; set off the length EC on DF; a line drawn through CF will be parallel to AB.

Problem 4.

To draw a line parallel to a given line AB *at a given distance from it.*

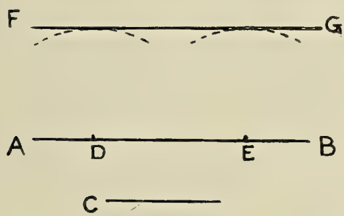


Fig. 39.

Let the length of the line C represent the given distance. Take any points D and E in line AB as centres, and with radius C describe arcs as shown; draw the line FG as a tangent to these arcs. FG will be parallel to AB.

Problem 5.

From a point C *in a given line* AB, *to draw a line perpendicular to* AB.

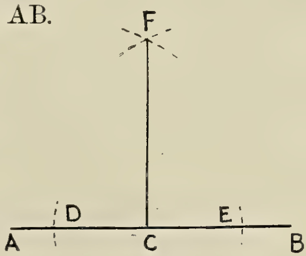


Fig. 40.

At the point C, with any radius, describe arcs cutting AB in D and E; with D and E as centres, and with the same radius, draw arcs intersecting at F; join FC, which will be perpendicular to AB.

Problem 6.

When the point is at, or near, the end of the given line AB.

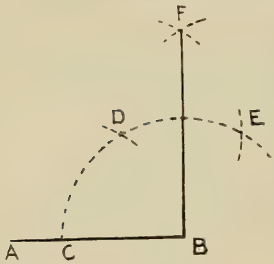


Fig. 41.

With B as centre, and with any radius BC, draw the arc CDE; and with the same radius, starting at C, set off arcs D and E; with each of these two points as centres and with any radius, draw arcs cutting each other at F; join FB, which will be perpendicular to AB.

Problem 7.

To draw a line perpendicular to a given line, from a point which is without the line.

Let AB be the given line and C the point.

With C as centre, and with any radius greater than CD, draw arcs cutting the line AB in E and F; from these points as centres, with any radius, describe arcs intersecting at G; join CG, which will be perpendicular to AB.

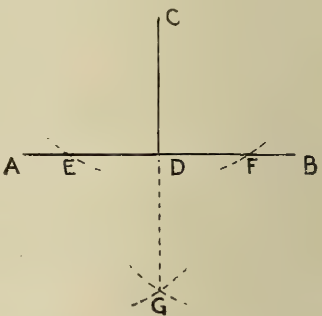


Fig. 42.

Problem 8.

When the point is opposite, or nearly opposite, one end of the line AB.

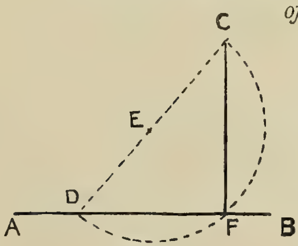


Fig. 43.

Let C be the given point. Take any point D on AB, not opposite the point C, and join CD and bisect it in E; with E as centre, and EC as radius, draw the semicircle CFD; join CF, which will be perpendicular to AB.

Problem 9.

To divide a given line AB *into any number of equal parts.*
Take five for example.

From one extremity A, draw the line AD, at any angle to AB; and from the end B, draw another line BC, parallel to AD. With any convenient radius, set off along the line AD, commencing at end A, the number of parts, less one, into which it is required to divide the given line; repeat the same operation on line BC, commencing at end B, with same radius; then join the points as shown, and the given line AB will be divided into five equal parts.

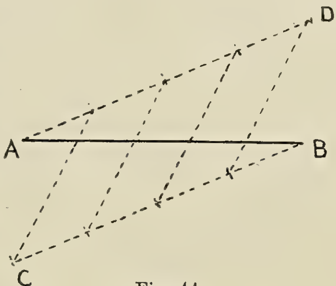


Fig. 44.

Problem 10.

Another Method.

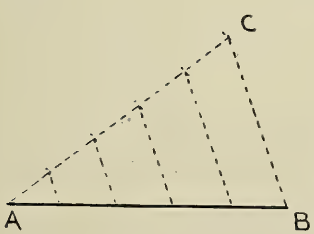


Fig. 45.

Draw the line AC at any angle to AB; AC may be of any convenient length. With any radius, mark off along AC the number of equal parts required; join the end of the last division C with B; draw lines from all the other points parallel to CB (Prob. 4), till they meet the line AB, which will then be divided as required.

Problem 11.

To divide a line proportionally to a given divided line.

Let AB be a given divided line. Take any point C and

join it with each end of AB; divide BC into four equal parts, and draw the lines D, E, and F parallel to AB; join the divisions on AB with C, then the divisions on the lines D, E, and F will represent respectively the proportions of $\frac{3}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$ of the divisions on the given line AB. If we wish to enlarge the divisions on AB to $1\frac{1}{4}$, for instance, produce the line CB and repeat one of the four divisions below B, which will give H; draw GH parallel to AB, and produce the divisions as shown.

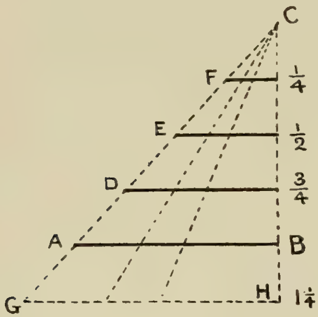


Fig. 46.

Problem 12.

From a given point B, *on a given line* AB, *to construct an angle equal to a given angle* C.

With point C of the given angle as centre, and with any radius, draw the arc EF; and with the same radius, with B as centre, draw the arc GH; take the length of the arc EF, and set it off on GH; draw the line BD through H. Then the angle GBH will be equal to the given angle C.

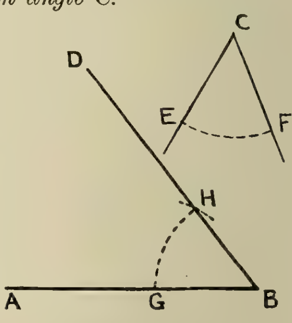


Fig. 47.

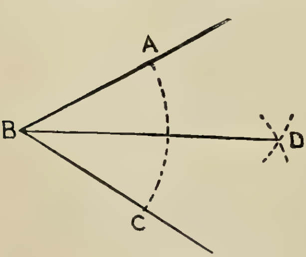


Fig. 48.

Problem 13.

To bisect a given angle ABC.

With B as centre, and with any radius, draw the arc AC; with A and C as centres, with any radius, draw the arcs intersecting at D; join DB, which will bisect the angle ABC.

Problem 14.

To trisect a right angle.

Let ABC be the right angle. With B as centre, and with any radius, draw the arc AC; with the same radius, and A and C as centres, draw the arcs E and D; join EB and DB, which will trisect the angle ABC.

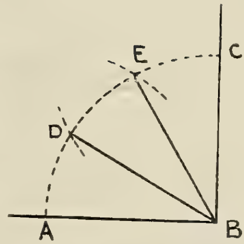


Fig. 49.

Problem 15.

To trisect any angle ABC.[1]

From B, with any radius, describe the arc AHC; bisect the angle ABC; join A and C; with D as centre, and with DA as radius, describe the semicircle AGC; and with the same radius, describe the arcs E and F from A and C; join AG; take the length AG and set it off from H along the line GB, which will give the point I; join EI and FI, which will give the points J and K on the arc AHC; join J and K with B, which will trisect the angle ABC.

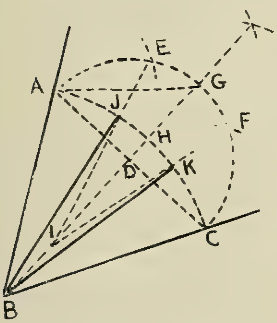


Fig. 50.

TRIANGLES.

Problem 16.

To construct an equilateral triangle on a given line AB.

With A and B as centres, and with AB as radius, describe arcs cutting each other at C; join C with A and B. Then ABC will be an equilateral triangle.

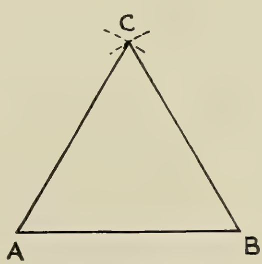


Fig. 51.

[1] This is one of the impossibilities of geometry; but this problem, devised by the author, gives an approximation so near, that the difference is imperceptible in ordinary geometrical drawing.

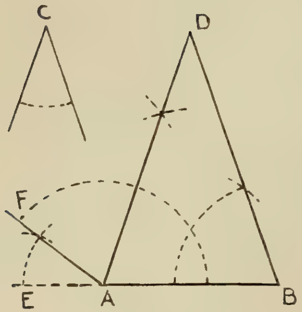


Fig. 52.

Problem 17.

On a given base AB *to construct an isosceles triangle, the angle at vertex to be equal to given angle* C.

Produce the base AB to E, and at A construct an angle making with AE an angle equal to C. Bisect the angle FAB by the line AD. From B draw a line making with AB an angle equal to DAB, and meeting AD in D. ADB will be the isosceles triangle required.

Problem 18.

On a given base AB, *to construct an isosceles triangle, its altitude to be equal to a given line* CD.

Bisect AB at E, and erect a perpendicular equal in height to the given line CD; join AF and BF, then AFB will be the isosceles triangle required.

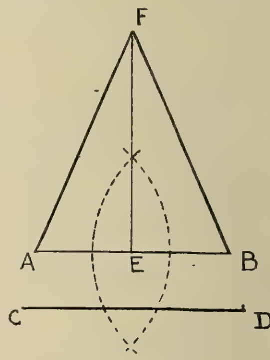


Fig. 53.

Problem 19.

To construct a triangle, the three sides A, B, *and* C *being given.*

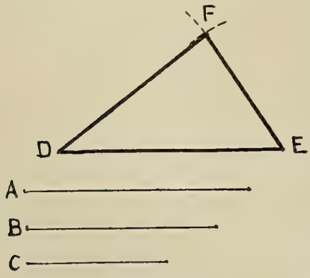


Fig. 54.

Draw the base DE equal to the given line A. From D as centre, and with radius equal to line B, describe an arc at F, and from E as centre, and with radius equal to line C, draw another arc, cutting the other at F; join FD and FE, which will give the triangle required.

Problem 20.

To construct a triangle with two sides equal to given lines A *and* B, *and the included angle equal to* C.

Make an angle DEF equal to given angle C, in required position. Mark off EF equal to line A, and ED equal to line B; join DF.

DEF is the triangle required.

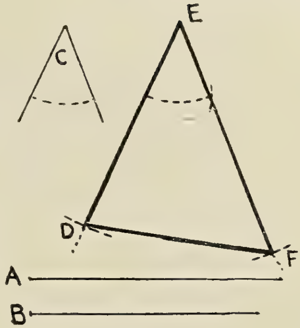


Fig. 55.

Problem 21.

To construct a triangle with a perpendicular height equal to AB, *and the two sides forming the vertex equal to the given lines* C *and* D.

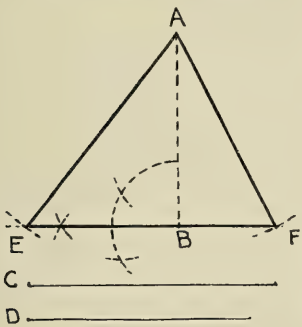


Fig. 56.

Through B draw the line EF at right angles to AB. From A as centre, and with radii equal to the lengths of the lines C and D respectively, draw arcs cutting the line EF; join AE and AF, which will give the triangle required.

Problem 22.

To construct a triangle, the base AB *being given, one angle of which is equal to* C, *and the difference of the sides equal to given line* D.

At end A of the base, construct an angle equal to the given angle C. Cut off AF equal to line D, the given difference of the sides; join FB. Bisect FB at right angles by a line meeting AF in E; join EB.

AEB is the required triangle.

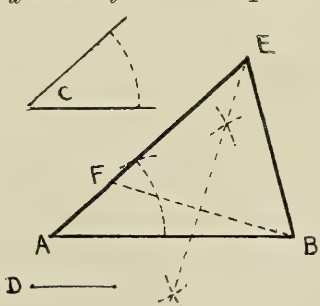


Fig. 57.

Problem 23.

To construct a triangle on a base equal to given line A, *with vertical angle equal to* D, *and sum of the two remaining sides equal to* BE.

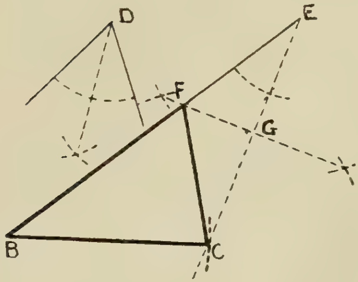


Fig. 58.

Draw the line BE, and at E construct an angle making with BE an angle equal to half the given angle D. From point B, with radius equal to given line A, draw an arc cutting EC at C; join BC. Bisect CE at right angles by line FG; join FC.

BCF will then be the triangle required.

Problem 24.

Construct a triangle with two sides equal to the given lines A *and* B *respectively, and the included median equal to given line C.*

Draw a triangle with the side DE equal to given line A, the side DF equal to given line B, and the third side FE equal to twice the given median C. Bisect the line FE at G. Draw DG and produce it to H; make GH equal to DG; join EH.

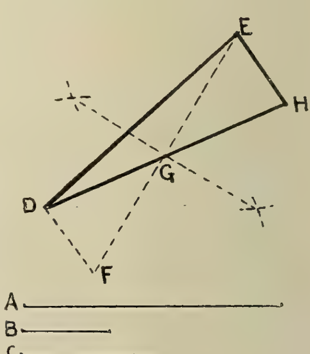


Fig. 59.

DEH will be the triangle required.

PROBLEM 25.

On a given base AB *to describe a triangle similar to a given triangle* DEF.

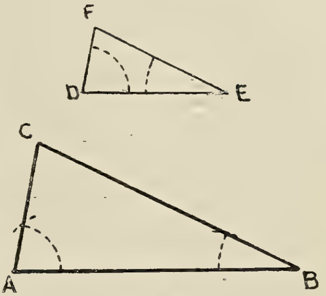


Fig. 60.

Make angles at A and B equal respectively to the angles at D and E. Produce the lines to meet at C. Then ABC will be the triangle required.

QUADRILATERALS.

PROBLEM 26.

To construct a square on a given base AB.

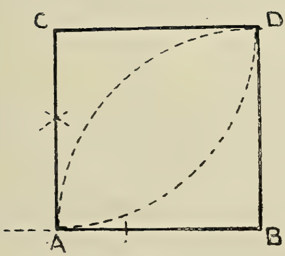


Fig. 61.

At point A erect a perpendicular to AB and equal to it. With B and C as centres, and radius equal to AB, draw intersecting arcs meeting at D; join DC and DB.

CABD is the square required.

PROBLEM 27.

To construct a square on a given diagonal AB.

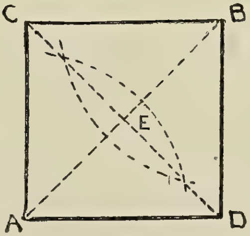


Fig. 62.

Bisect AB with the line CD at right angles to it. Mark off EC and ED equal to EA and EB; join CA and AD, and CB and BD.

CADB is the square required.

Problem 28.

To construct a rectangle with sides equal to given lines A *and* B.

Draw the line CD equal to given line A. At D erect a perpendicular DF equal to given line B. With C as centre

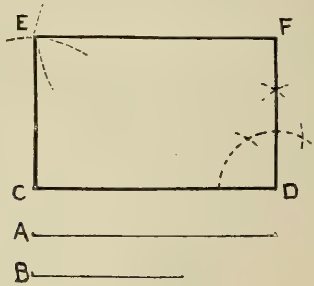


Fig. 63.

and radius equal to line B, and with F as centre and radius equal to line A, draw arcs intersecting each other at E; join EC and EF, which will give the rectangle required.

Problem 29.

To construct a rectangle with diagonal equal to given line A, *and one side equal to given line* B.

Draw the line CD equal to given line A. Bisect CD at E.

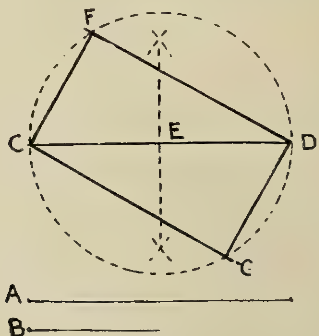


Fig. 64.

With E as centre, and with radius EC, draw a circle; from C

and D as centres, and with given line B as radius, set off the points F and G; join CG, GD, DF, and FC.

CGDF is the required rectangle.

Problem 30.

To construct a rhombus with sides equal to given line A, *and angle equal to given angle* C.

Make the base DE equal to given line A. At D construct an angle equal to given angle C. Set off DF equal to DE; with F and E as centres, and radius equal to DE, draw arcs intersecting at G; join FG and EG.

DEGF will be the required rhombus.

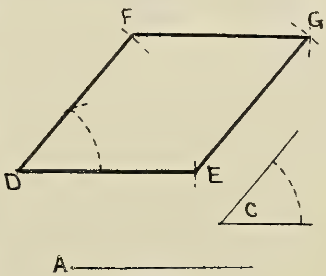


Fig. 65.

CONVERGENT LINES; CIRCLES AND ARCS.

Problem 31.

To draw a line bisecting the angle between two given converging lines AB *and* CD, *when the angular point is inaccessible.*

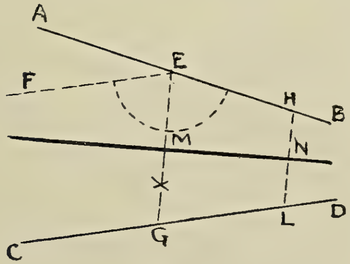


Fig. 66.

From any point E in AB, draw a line EF parallel to CD. Bisect the angle BEF by the line EG. At any point H between E and B, draw HL parallel to EG. Bisect EG and HL in M and N. Join MN, which, produced, is the bisecting line required.

Problem 32.

Through the given point A, to draw a line which would, if produced, meet at the same point as the given lines BC and DE produced.

Draw any convenient line FG; join AF and AG. Draw any line HK parallel to FG. At H draw the line HL parallel

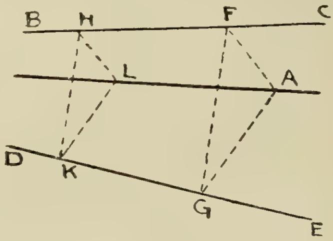


Fig. 67.

to FA, and at K draw the line KL parallel to GA, cutting each other at L. Draw a line through L and A; AL is the convergent line required.

Problem 33.

To find the centre of a circle.

Draw any chord AB, and bisect it by a perpendicular DE,

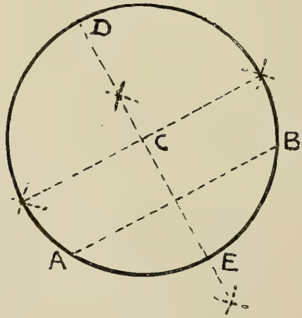


Fig. 68.

which will be a diameter of the circle. Bisect DE in C, which will be the centre of the circle.

Problem 34.

To draw a circle through three given points A, B, C.

Join AB and BC. Bisect AB and BC by perpendiculars cutting each other at D, which will be the centre of the circle. From D as centre, and DA as radius, describe a circle, which will then pass through the given points A, B, C, as required.

Note.—The two following problems are constructed in the same manner.

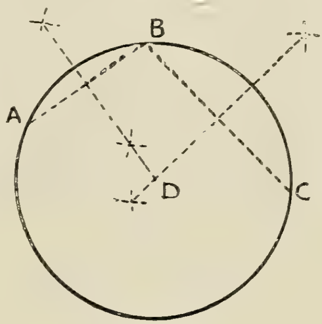


Fig. 69.

To draw the arc of a circle through three given points A, B, C.

To find the centre of a circle from a given arc AC.

Problem 35.

At the given equidistant points A, B, C, D *on a given arc, to draw a number of radial lines, the centre of the circle being inaccessible.*

On the points A, B, C, D, etc., as centres, with radii larger than a division, describe arcs cutting each other at EF, etc. Thus from A and C as centres, describe arcs cutting each other at E, and so on. Draw the lines BE, CF, etc., which will be the radial lines required.

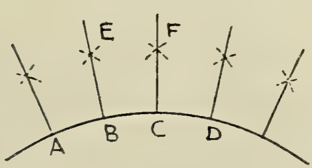


Fig. 70.

Problem 36.

To draw the arc of a circle through three given points A, B, C, *the centre of the circle being inaccessible.*

With A and C as centres, and with a radius equal to AC,

draw indefinite arcs. From the points A and C draw lines through B till they meet the arcs in D and E. From D and E set off short equal distances on the arcs above and below; join the divisions on arc AD to point C, and the divisions on arc CE to point A. Where the lines from corresponding points intersect, we obtain a point in the arc; for instance, where the line from E1 intersects the line from D1, we get the point F; and so on with the other points, by joining which with a fair curve, we get the required arc.

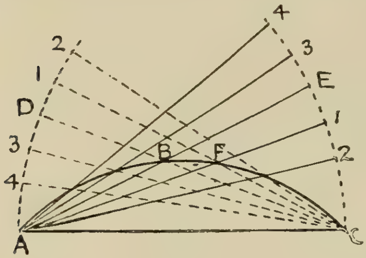


Fig. 71.

GEOMETRICAL PATTERN DRAWING.

These designs are based on the equilateral triangle and square; they should be copied to a much larger scale than shown.

Figs. 72, 73, 74, and 75, are drawn with the 60° set-square, worked against the edge of the tee-square. All three angles of the set-square are used, viz. 90°, 60°, and 30°.

The remaining figures are drawn with the 45° set-square.

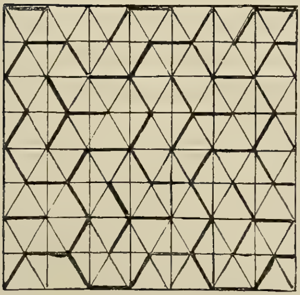


Fig. 72.

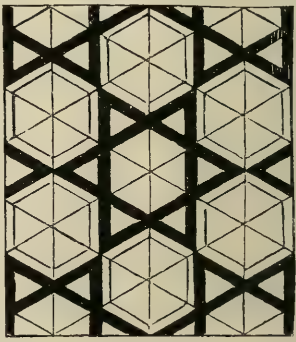


Fig. 73.

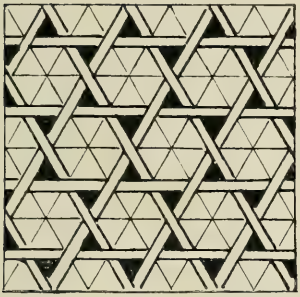


Fig. 74.

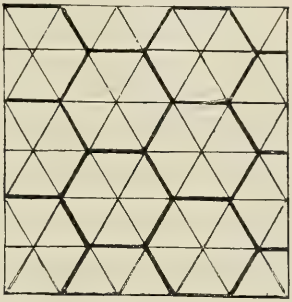


Fig. 75.

Fig. 76.

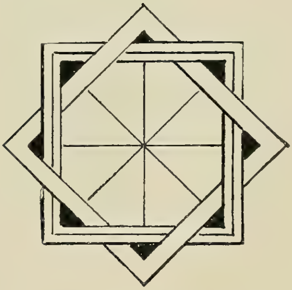


Fig. 77.

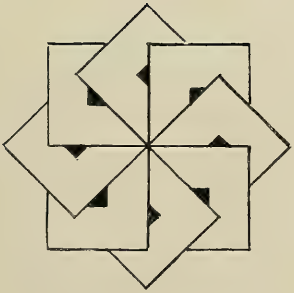


Fig. 78.

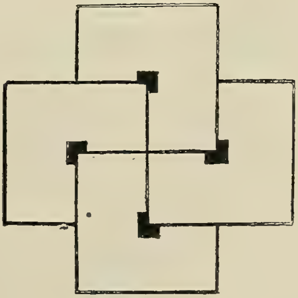


Fig. 79.

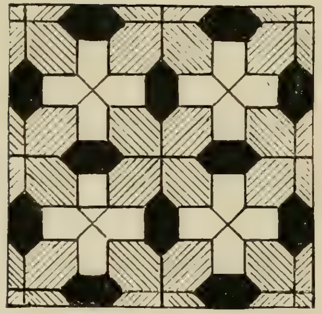


Fig. 80.

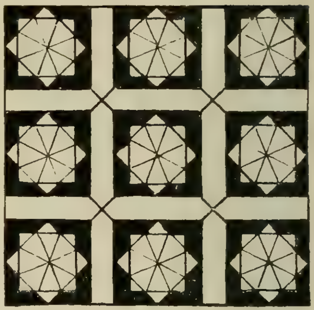


Fig. 81.

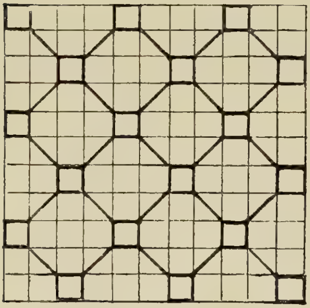


Fig. 82.

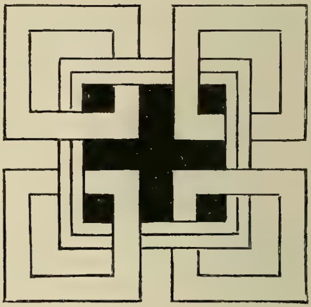
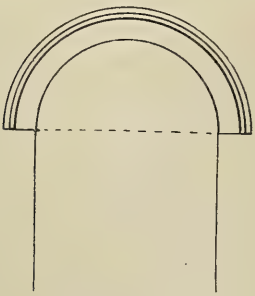


Fig. 83.

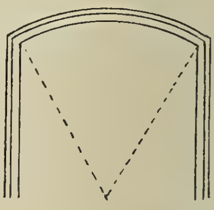
Arches.

These figures show the application of arcs of circles to the formation of the different kinds of arches.



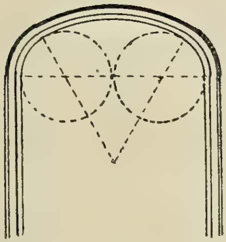
Semicircular

Fig. 84.



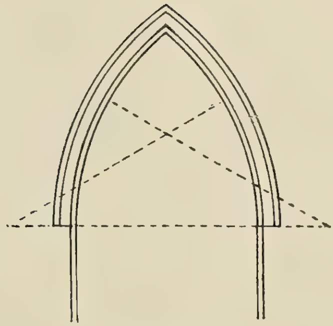
Segmental

Fig. 85.



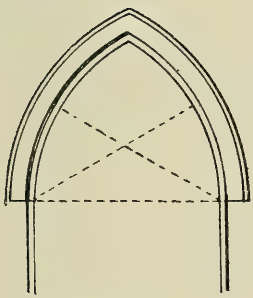
Elliptic.

Fig. 86.



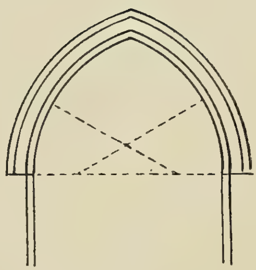
Lancet

Fig. 87.



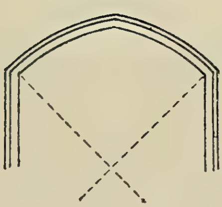
Equilateral

Fig. 88.



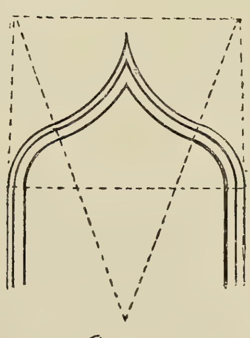
Drop

Fig. 89.



Segmental Pointed.

Fig. 90.



Ogee.

Fig. 91.

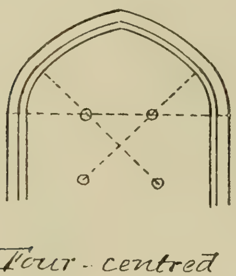


Fig. 92.

Figs. 84, 85, and 86 represent the forms of arches used before the introduction of Gothic or pointed architecture.

Figs. 87, 88, 89, and 90 belong to the earlier periods of Gothic, while Figs. 91 and 92 represent those of a later period.

CHAPTER III

POLYGONS

Regular Polygons are figures that have equal sides and equal angles. To construct a regular polygon, we must have the length of one side and the number of sides; if it is to be inscribed in a circle, the number of its sides will determine their length and the angle they form with each other.

If we take any polygon, regular or irregular, and produce

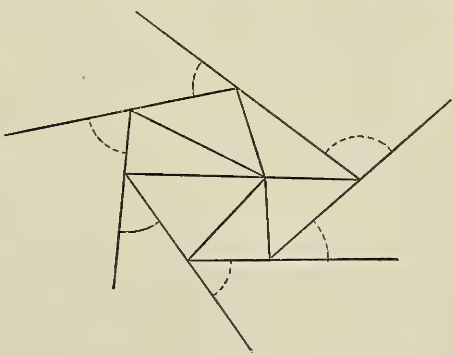


Fig. 93.

all its sides in one direction only, Fig. 93, we shall find that the total of all the exterior angles, shown by the dotted curves, is equal to 360°, or four right angles; and if we join each angle of the polygon to any point in its centre, the sum of the angles at this point will also be 360°, and there will be as many angles formed in the centre as there are exterior angles.

In regular polygons these angles at the centre will, of course,

be equal to each other; and if we produce the sides in one direction, as in Fig. 93, the exterior angles will be equal to each other; and as the number of angles at the centre is equal to the number of exterior angles, and the sum of the angles in each instance is equal, the angle at the centre must equal the exterior angle.

To construct any regular polygon on the given line AB, for example, a *nonagon*, $360° \div 9 = 40°$. So if we draw a line at B, making an angle of $40°$ with AB produced, it will give us the exterior angle of the nonagon, from which it will be easy to complete the polygon.

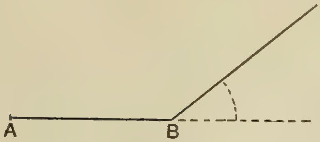


Fig. 94.

The perimeter of a polygon is sometimes given, *e.g. Construct an octagon the perimeter of which is* 6 *inches.*

$$\underset{\text{inches.}}{6} \div 8 = \underset{\text{inches.}}{\cdot 75} = \underset{\text{inches.}}{\tfrac{3}{4}}$$

Draw the line AB this length. It has been shown that the exterior angles and those at the centre of a regular polygon are equal, $\therefore\ 360° \div 8 = 45°$. Produce the line AB, and construct an angle of $45°$. Make BC = AB. We now have three points from which we can draw the circle containing the required polygon (Prob. 34).

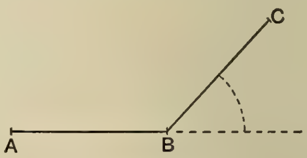


Fig. 95.

The polygon could also be drawn, after finding the length of AB, by any of the methods shown for constructing a polygon on a given straight line.

The centre of any regular polygon is the centre of the circle that circumscribes it.

Any regular polygon can be inscribed in a circle by setting off the angles at the centre.

If tangents to a circle circumscribing a regular polygon be drawn parallel to the sides of the inscribed polygon, a similar figure will be described about the circle; or if tangents be drawn at the angles of the inscribed polygon, the circle will also be "described by," *i.e.* contained in, a similar figure.

In the following problems two *general* methods are given for constructing regular polygons on a given line, and two for inscribing them in a given circle; but as these general methods require either a line or an arc to be first divided into equal divisions, the special methods for individual polygons are preferable, which are also given.

See also how to construct any angle without a protractor (Prob. 169), and its application to polygons.

Problem 37.

To inscribe in a circle, a triangle, square, pentagon, hexagon, octagon, decagon, or duodecagon.

Describe a circle, and draw the two diameters AE and BD at right angles to each other; join BA. Set off on circumference, AF equal to AC; join AF and EF. With D as centre, and radius equal to EF, draw the arc G on EA produced. With G as centre, and radius equal to AC, set off H on circumference; join AH. With D and E as centres, and radius equal to DC, set off the arcs I and J on circumference; join ID. With CG as radius, and J as centre, mark off K on diameter BD. With CK as radius, mark off on circumference from B the arcs L and M; join BL and BM; then :—

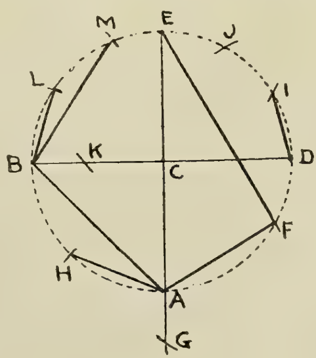


Fig. 96.

EF	will equal	one side of	an *equilateral triangle.*
AB	,,	,,	a *square.*
BM	,,	,,	*pentagon.*
AF	,,	,,	*hexagon.*
AH	,,	,,	*octagon.*
BL	,,	,,	*decagon.*
DI	,,	,,	*duodecagon.*

Problem 38.

To inscribe any regular polygon in a given circle.
A *heptagon,* for example.

Describe a circle, and draw the diameter AB. Divide AB

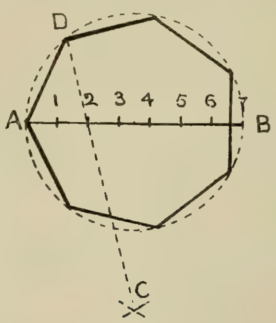


Fig. 97.

into as many equal divisions as there are sides to the polygon (in this instance seven). With A and B as centres, and with radius equal to AB, describe intersecting arcs at C. From C draw the line CD, passing through the second division from A,[1] till it meets the circumference at D. Join AD, which will give one side of the polygon; to complete it, mark off AD round the circumference.

This method of constructing polygons is due to the Chevalier Antoine de Ville (1628), and although useful for practical purposes, is not mathematically correct.

[1] Whatever number of sides the polygon may have, the line CD is always drawn through the second division from A.

Problem 39.

To inscribe any regular polygon in a given circle (second method). For example, a *nonagon.*

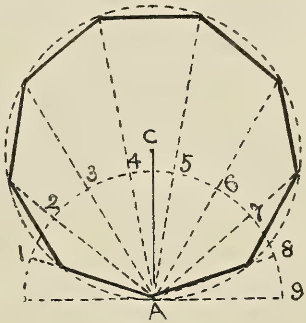


Fig. 98.

Describe a circle, and draw the radius CA. At A draw a tangent to the circle. With A as centre, and with any radius, draw a semicircle, and divide it into as many equal parts as there are sides to the polygon (in this instance nine).

From point A draw lines through each of these divisions till they meet the circumference. Join these points, which will give the polygon required.

Problem 40.

On a given line AB, *to describe a regular polygon.* For example, a *heptagon.*

Produce AB to C. With A as centre, and AB as radius, describe a semicircle, and divide it into as many equal divisions as there are sides to the polygon (in this instance seven). Join A with the second division from C in the semicircle,[1] which will give point D. Join AD. Through the points DAB describe a circle. Set off the distance AD round the circumference, and join the points marked, which will give the required polygon.

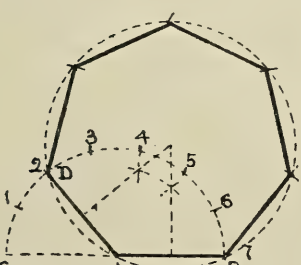


Fig. 99.

[1] The line AD is always drawn to the second division from C, whatever number of sides the polygon may contain.

PROBLEM 41.

On a given line AB, *to describe a regular polygon* (second method). For example, a *pentagon.*

At point B erect a perpendicular equal to AB. With B as centre, and radius BA, draw the quadrant AC, and divide it into as many equal divisions as there are sides to the polygon (in this case five).

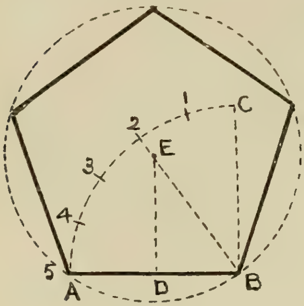


Fig. 100.

Join the point B with the second division from C.[1] Bisect AB at D, and erect a perpendicular till it meets the line drawn from B to the second division, which will give point E. From E as centre, and with radius EB, describe a circle. From point A, with distance AB, mark off round the circumference the points of the polygon. Join these points, which will give the pentagon required.

PROBLEM 42.

On a given straight line AB, *to construct a regular pentagon.*

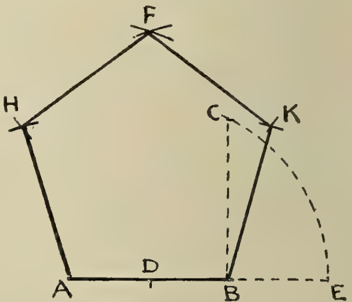


Fig. 101.

At B erect BC perpendicular to AB, and equal to it. Bisect AB in D. With D as centre, and DC as radius, draw the arc CE meeting AB produced in E. With A and B as centres, and radius equal to AE, draw intersecting arcs at F. With A, B, and F as centres, and radius equal to AB, draw intersecting arcs at H and K. Join AH, HF,

[1] See note to previous problem.

FK, and KB. Then AHFKB will be the pentagon required.

Problem 43.

On a given straight line AB, *to construct a regular hexagon.*

With A and B as centres, and radius AB, draw the arcs intersecting each other at C. With C as centre, and with the same radius, draw a circle. With the same radius, commencing at A, set off round the circle the points D, E, F, G. Join AD, DE, EF, FG, and GB, which will give the hexagon required.

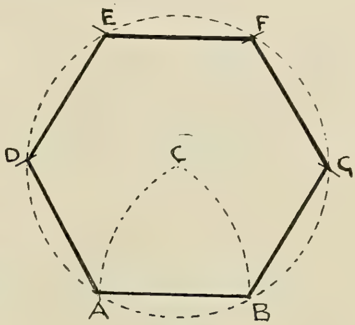


Fig. 102.

Problem 44.

In a given circle to inscribe a regular heptagon.

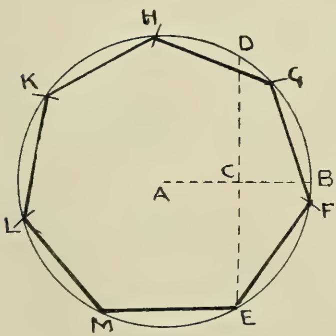


Fig. 103.

Draw any radius AB, and bisect it in C. Through C draw DE perpendicular to AB. With CD as radius, commencing at E, set off round the circle the points F, G, H, K, L, and M, by joining which we get the required heptagon.

Problem 45.

On a given line AB, *to construct a regular heptagon.*

With B as centre, and BA as radius, draw a semicircle

meeting AB produced in C. With centre A, and radius AB, draw an arc cutting the semicircle in D. Draw DE perpendicular to AB. With C as centre, and radius equal to DE, draw an arc cutting the semicircle in F. Join BF. From the three points A, B, F, find the centre of the circle H (Prob. 34). With H as centre, and radius HA, draw a circle. With AB as radius, commencing at F, set off on the circle the points K, L, M, and N, by joining which we get the heptagon required.

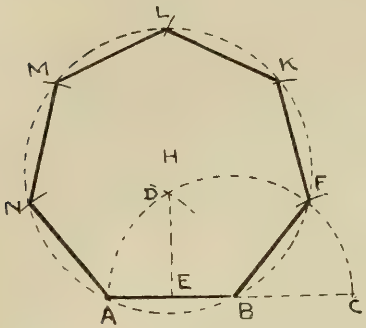


Fig. 104.

Problem 46.

On a given line AB, *to construct a regular octagon.*

At A and B erect the perpendiculars AC and BD. Produce AB to E. Bisect the angle DBE by the line BF. Make BF equal to AB. From F draw the line FH parallel to AB, and make KH equal to LF. Join AH. Set off on AC and BD, from points A and B, a length equal to HF, which will give the points C and D. With the points C, D, H, and F as centres, and a radius equal to AB, draw intersecting arcs at M and N. Join HM, MC, CD, DN, and NF, which will give the octagon required.

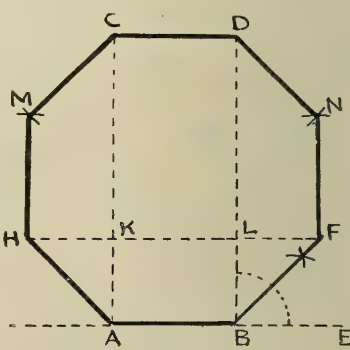


Fig. 105.

Problem 47.

In a given circle to inscribe a nonagon.

Draw the diameters AB and CD perpendicular to each other. With A as centre, and radius AE, draw the arc cutting the circle in F. With B as centre, and radius BF, draw the arc cutting CD produced in G. With G as centre, and radius GA, draw the arc cutting CD in H. With HC as radius, commencing at A, set off on the circle the points K, L, M, N, O, P, Q, and R, by joining which we get the nonagon required.

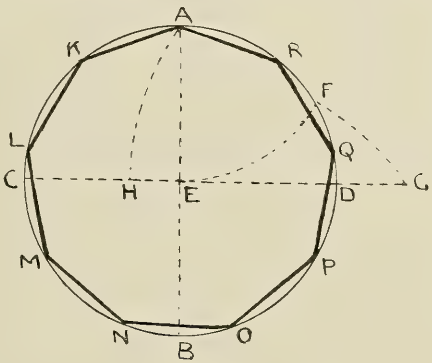


Fig. 106.

Problem 48.

In a given circle to inscribe a regular undecagon.

Draw the two diameters AB and CD cutting each other in E. With A as centre, and AE as radius, draw an arc cutting the circle in H. With D as centre, and DE as radius, draw an arc cutting the circle in F. With F as centre, and FH as radius, draw the arc cutting CD in K. Join HK. Then HK is equal to one side of the undecagon. With HK as radius, starting from H, set off on the circle the points of the required polygon, and join them.

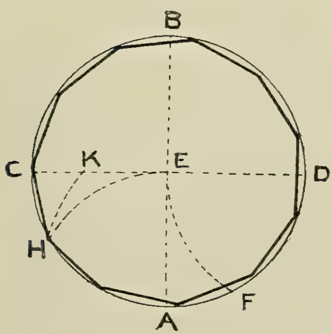


Fig. 107.

CHAPTER IV

INSCRIBED AND DESCRIBED FIGURES

Problem 49.

To inscribe an equilateral triangle in a given circle ABC.

Find the centre E (Prob. 33), and draw the diameter DC. With D as centre, and DE as radius, mark off the points A and B on the circumference of the circle. Join AB, BC, and CA. Then ABC will be the inscribed equilateral triangle required.

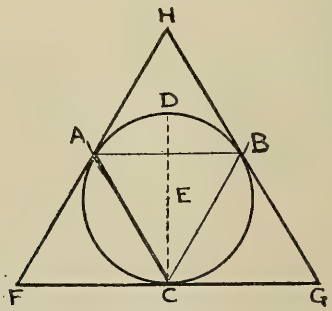


Fig. 108.

Problem 50.

To describe an equilateral triangle about a given circle ABC.

At the points A, B, and C draw tangents to the circle (Prob. 84), and produce them till they meet in the points F, G, and H. Then the equilateral triangle FGH will be described about the circle ABC, as required.

PROBLEM 51.

To inscribe a circle in a given triangle ABC.

Bisect the angles CAB and ABC by lines meeting in D. From D let fall the line DE, perpendicular to AB. With D as centre, and DE as radius, inscribe the circle required.

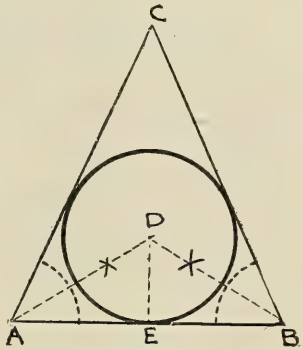


Fig. 109.

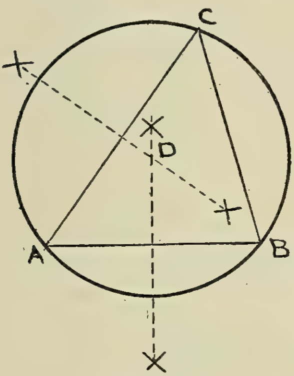


Fig. 110.

PROBLEM 52.

To describe a circle about a given triangle ABC.

Bisect the two sides AB and AC by lines meeting in D. With D as centre, and DA as radius, describe the required circle.

PROBLEM 53.

To describe an equilateral triangle about a given square ABCD.

With C and D as centres, and with CA as radius, describe arcs cutting each other at E. With E as centre, and with the same radius, mark off the points F and G on these arcs. Join CF and DG, and produce them till they meet in the point H, and the base AB produced in K and L. Then HKL is the required equilateral triangle.

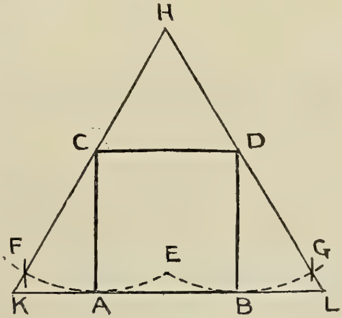


Fig. 111.

Problem 54.

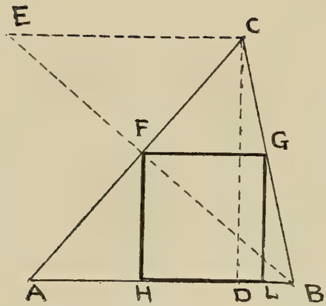


Fig. 112.

To inscribe a square in a given triangle ABC.

From C draw CD perpendicular to AB, and CE perpendicular to CD, and equal to it. Draw the line EB cutting AC in F. Draw FG parallel to AB. Draw the lines FH and GL perpendicular to AB. Then FGLH is the square required.

Problem 55.

In a given triangle ABC, *to inscribe an oblong having one of its sides equal to the given line* D.

From A, along the base AB, set off AE equal to the given line D. From E draw the line EF parallel to AC. Through F draw the line FG parallel to the base AB. From G and F draw the lines GH and FL perpendicular to AB.

GFLH is the oblong required.

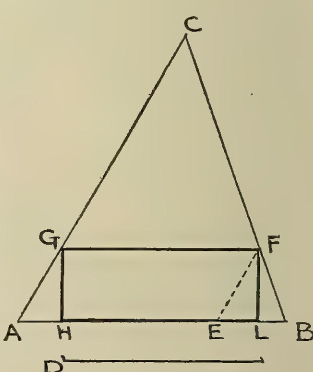


Fig. 113.

Problem 56.

To inscribe a square in a given circle.

Draw any two diameters AB and CD at right angles to each other. Join the extremities of these diameters.

ACBD is the inscribed square required.

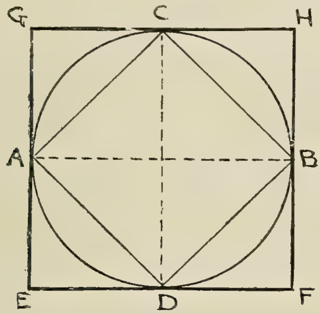


Fig. 114.

Problem 57.

To describe a square about a given circle.

At the points A, C, B, D draw tangents meeting each other at the points E, F, G, H (Prob. 84).

EFHG is the described square required.

Problem 58.

To inscribe a circle in a given square.

Draw the diagonals AB and CD intersecting each other at E. From E draw EF perpendicular to AD. With E as centre, and EF as radius, draw a circle. This will be the inscribed circle required.

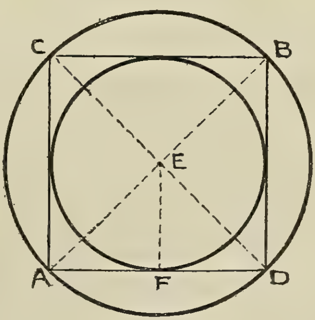


Fig. 115.

Problem 59.

To describe a circle about a given square.

With centre E, and radius EA, draw a circle. This will be the described circle required.

PROBLEM 60.

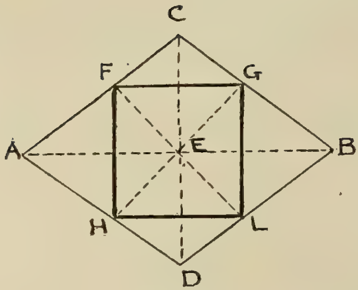


Fig. 116.

To inscribe a square in a given rhombus.

Draw the two diagonals AB and CD. Bisect the angles AEC, AED, and produce the lines each way till they meet the sides of the rhombus in the points F, G, H, L. Join FG, GL, LH, and HF.

FGHL is the inscribed square required.

PROBLEM 61.

To inscribe a circle in a given rhombus.

Draw the diagonals AB and CD intersecting each other in E. From E draw the line EF perpendicular to AD (Prob. 6). With E as centre, and EF as radius, draw a circle. This is the inscribed circle required.

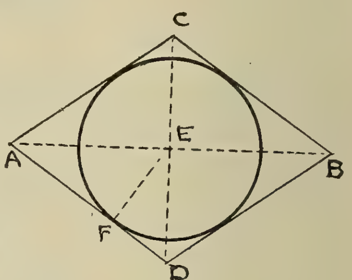


Fig. 117.

PROBLEM 62.

To inscribe an equilateral triangle in a given square ABCD.

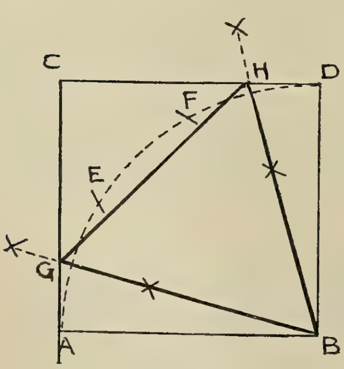


Fig. 118.

With B as centre, and radius BA, draw the quadrant AD; and with same radius, with A and D as centres, set off on the quadrant the points F and E. Bisect AE and FD, and through the points of bisection draw lines GB and HB, cutting the given square in G and H. Join BG, GH, and HB.

BGH is the inscribed equilateral triangle required.

Problem 63.

To inscribe an isosceles triangle in a given square ABCD, *having a base equal to the given line* E.

Draw the diagonal AD. From A, along AD, set off AF

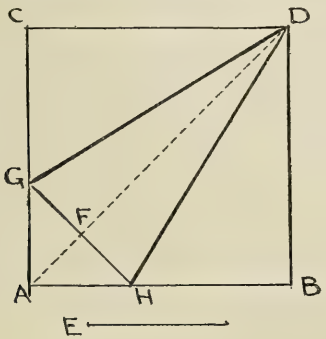


Fig. 119.

equal to half the given base E. Through F draw the line GH perpendicular to AD (Prob. 5). Join GD and HD.

GDH is the inscribed isosceles triangle required.

Problem 64.

To inscribe a square in a given trapezium ABCD *which has its adjacent pairs of sides equal.*

Draw the two diagonals AB and CD. From point C set off CE perpendicular and equal to CD. Join EA by a line cutting CB in G. Draw the line GF parallel to AB. From points F and G draw the lines FH and GK parallel to CD. Join HK.

FGKH is the inscribed square required.

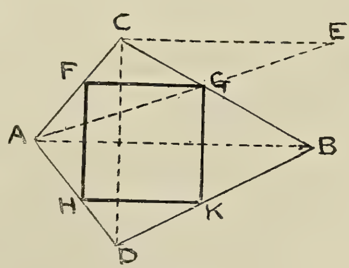


Fig. 120.

Problem 65.

To inscribe a circle in a given trapezium ACBD *which has its adjacent pairs of sides equal.*

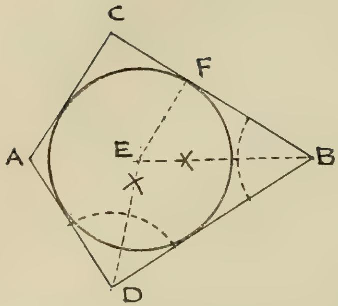


Fig. 121.

Bisect any two adjacent angles, as ADB and DBC, by lines meeting in E. From E draw EF perpendicular to CB. With E as centre, and EF as radius, draw a circle. This will be the inscribed circle required.

Problem 66.

To inscribe a rhombus in a given rhomboid ABCD.

Draw the two diagonals AD and CB intersecting each other in E. Bisect any two adjacent angles, as CED and DEB, by lines cutting the given rhomboid in F and G, and produce these lines to K and H. Join FG, GK, KH, and HF.

FGKH is the inscribed rhombus required.

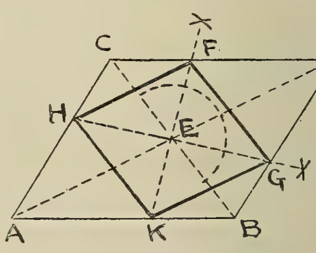


Fig. 122.

Problem 67.

To inscribe an octagon in a given square ABCD.

Draw the two diagonals AD and CB intersecting each

other in E. With A as centre, and AE as radius, mark off the points F and G on the sides of the square. Proceed in the same manner with the angles B, C, and D as centres, which will give the eight points required on the given square, by joining which we obtain the required inscribed octagon.

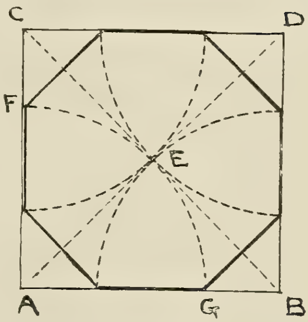


Fig. 123.

PROBLEM 68.

To inscribe a square in a given hexagon ABCDEF.

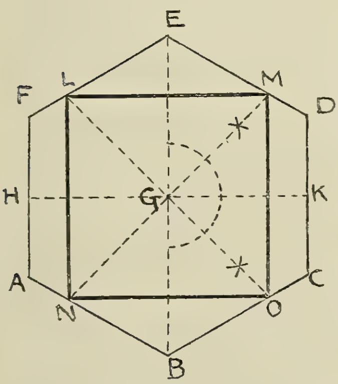


Fig. 124.

Draw the diagonal EB; bisect it in G, and draw the line HK perpendicular to it. Bisect any two adjacent angles, as BGK and EGK, by lines meeting the hexagon in M and O. Produce these two lines till they meet the opposite side of the hexagon in L and N. Join LM, MO, ON, and NL.

LMON is the inscribed square required.

PROBLEM 69.

To inscribe four equal circles in a given square ABCD; *each circle to touch two others, as well as two sides of the given square.*

Draw the two diagonals AD and CB intersecting each other

in the point E. Bisect the sides of the square in the points F, G, H, and L (Prob. 1). Draw the lines FH and GL. Join FG, GH, HL, and LF, which will give the points M, O, P, and R. From point M draw the line MN parallel to AB. With M, O, P, and R as centres, and radius equal to MN, describe circles. These will be the four inscribed circles required.

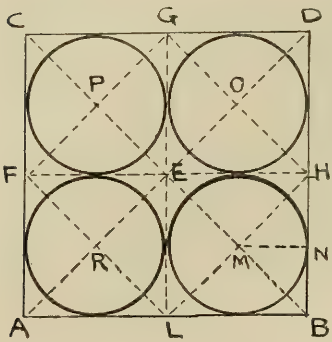


Fig. 125.

Problem 70.

To inscribe four equal circles in a given square ABCD ; *each circle to touch two others, and one side only of the given square.*

Draw the two diagonals AD and CB intersecting each other in the point E. Bisect the sides of the square in the points F, G, H, and L. Join FH and GL. Bisect the angle GDE by a line meeting GL in M. With centre E, and radius EM, mark off the points N, O, and P. With the points M, N, O, and P as centres, and radius equal to MG, describe the four inscribed circles required.

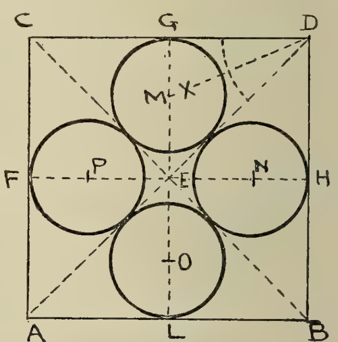


Fig. 126.

Problem 71.

To inscribe three equal circles in a given equilateral triangle ABC ; *each circle to touch the other two, as well as two sides of the given triangle.*

Bisect the two sides of the triangle by lines intersecting

each other at the centre D. Draw a line from point C through the centre D till it meets the base at E. Bisect the angle BEC by a line meeting DB in F. With D as centre, and DF as radius, mark off the points K and L. From F draw the line FH perpendicular to AB. With the points F, K, and L as centres, and with a radius equal to FH, draw the three inscribed circles, required.

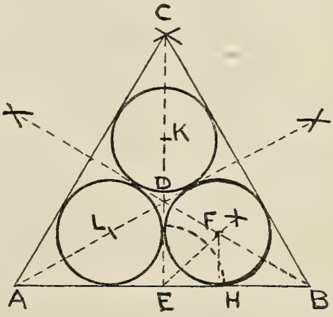


Fig. 127.

Problem 72.

In a given equilateral triangle ABC, *to inscribe three equal circles touching each other and one side of the triangle only.*

Bisect two sides of the triangle by lines intersecting each other at the point D (Prob. 1). Join CD and produce it to E. Bisect the angle DBE by a line cutting CE in F. With D as centre, and DF as radius, mark off the points G and H. From the points F, G, and H as centres, and with a radius equal to FE, draw the three inscribed circles required.

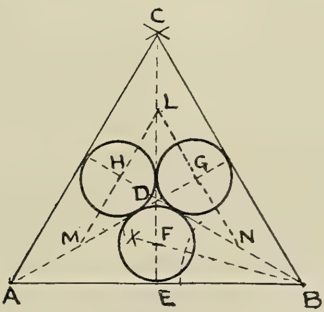


Fig. 128.

Problem 73.

In a given equilateral triangle ABC, to inscribe six equal circles touching each other and one side of the triangle only.

Having drawn the three circles according to the last problem, draw lines through G and H, parallel to the sides of the triangle, till they meet the lines bisecting the angles in the points L, M, and N. These points are the centres of the three circles which will complete the six inscribed circles required.

Problem 74.

In a given octagon to inscribe four equal circles touching each other.

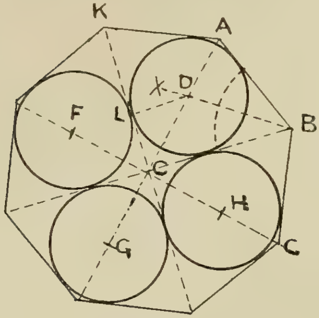


Fig. 129.

Draw the four diagonals. Bisect the angle ABC by a line intersecting AC in D. With C as centre, and CD as radius, mark off the points F, G, and H. From D draw the line DL perpendicular to KC. With D, F, G, and H as centres, and a radius equal to DL, draw the four inscribed circles required.

Problem 75.

In a given circle to draw four equal circles touching each other.

Find centre of circle E (Prob. 33). Draw the two diameters AB and CD at right angles to each other. At A and D draw tangents to the circle to meet at the point F (Prob. 84). Draw FE. Bisect the angle EFD by a line cutting CD in G. With E as centre, and EG as radius, mark off the points H, K, and L. With G, H, K, and L as centres, and with a radius equal to GD, draw the four inscribed circles required.

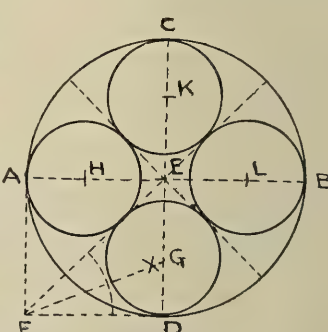


Fig. 130.

Problem 76.

In a given circle to inscribe any number of equal circles touching each other. For example, *five.*

Find the centre C (Prob. 33). Divide the circumference into five equal parts (Prob. 37), and draw the five radii to meet the circumference in the points M, N, O, P, and R. Bisect the angle MCN by a line meeting the circumference in A. Through A draw a line at right angles to CE till it meets the lines CM and CN produced in B and D. Bisect the angle DBC by a line to meet CA in E. With C as centre, and CE as radius, draw a circle, and bisect each arc on this circle, between the five radii, in the points F, G, H, and L (Prob. 2). With E, F, G, H, and L as centres, and a radius equal to EA, draw the five inscribed circles required.

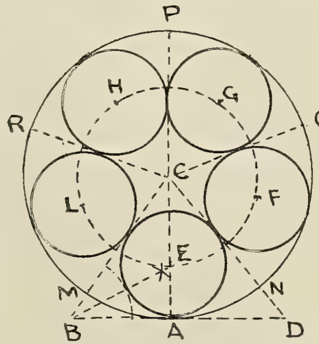


Fig. 131.

Problem 77.

About a given circle A, to describe six circles equal to it, touching each other as well as the given circle.

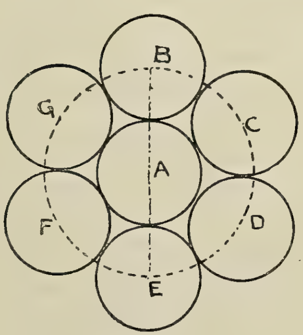


Fig. 132.

Find centre of circle A (Prob. 33). With A as centre, and a radius equal to the diameter of the given circle, draw the circle BCDEFG. Draw the diameter BE. Take the diameter of the given circle and mark off from E the points D, C, B, G, and F. With each of these points as a centre, and a radius equal to that of the given circle, draw the six circles required.

CHAPTER V

FOILED FIGURES

Foiled figures are constructed on the regular polygons, and are of two kinds: viz. those having *tangential arcs*, and those having *adjacent diameters.*

Problem 78 is an illustration of the former kind. The arcs simply touch, and if produced would not intersect each other. The angles of the polygon are the centres of the circles containing the arcs.

Problem 79 is an illustration of the latter kind. The sides of the polygon form the diameters of the semicircles, the centre of each side being the centre of the circle containing the arc. If these arcs were produced, they would intersect each other.

Problem 78.

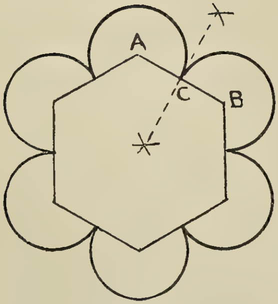


Fig. 133.

To construct a foiled figure about any regular polygon, having tangential arcs. For example, a hexagon.

THE HEXAFOIL.

Bisect one side AB in C (Prob. 1). With each of the angular points as centres, and AC as radius, draw the six arcs, as required.

Problem 79.

To construct a foiled figure about any regular polygon having adjacent diameters. For example, a pentagon.

THE CINQUEFOIL.

Bisect each side of the pentagon (Prob. 1) in the points C, D, E, F, and G. With each of these points as centres, and with a radius equal to CA, draw the five arcs required.

Note.—If these arcs are to have a stated radius, the length of the line AB, in each instance, will be twice the required radius.

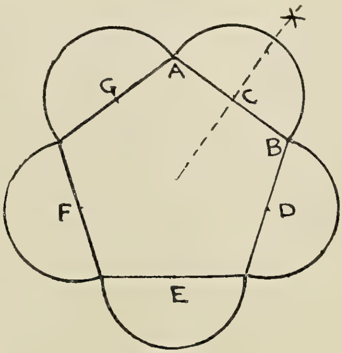


Fig. 134.

Problem 80.

THE TREFOIL.

In a given equilateral triangle ABC, *to inscribe a trefoil.*

Bisect the angles CAB and ABC by lines produced till they meet the sides of the triangle in D and E (Prob. 13). From C, through centre L, draw the line CF. Join DE, EF, and FD. From G draw GI perpendicular to AC (Prob. 4). With G, H, and K as centres, and a radius equal to GI, draw the three arcs till they meet each other, which will form the trefoil required.

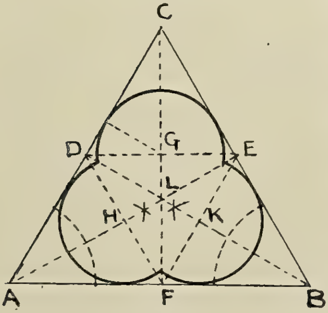


Fig. 135.

Problem 81.

Within a given circle to inscribe three equal semicircles having adjacent diameters.

Find centre of circle A (Prob. 33). Draw the diameter BC, and the radius AD perpendicular to it (Prob. 5). Trisect the angle BAD in E and F (Prob. 14). Set off DG equal to DF. Join FA and GA by lines produced to L and H. Join EG by a line cutting FL in M. With A as centre, and AM as radius, set off the points N and O. Join MN, NO, and OM, which are the diameters of the required semicircles. With R, P, and S as centres, and a radius equal to RM, draw the three semicircles required.

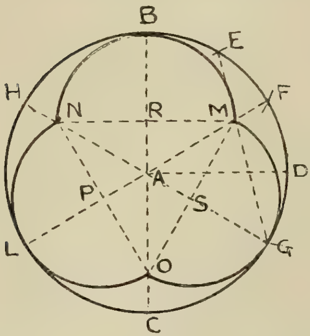


Fig. 136.

Problem 82.

THE QUATREFOIL.

In a given square ABCD, *to inscribe four semicircles having adjacent diameters.*

Draw the diagonals AD and CB. Bisect each side of the square in the points E, F, G, and H, and join EF and GH. Bisect HD in K and FB in L (Prob. 1), and join KL, cutting GH in N. With P as centre, and PN as radius, mark off the points M, O, and R, and join NM, MO, OR, and RN, which will form the diameters of the required semicircles. With S, T, U, and V as centres, and with a radius equal to SN, draw the four semicircles required.

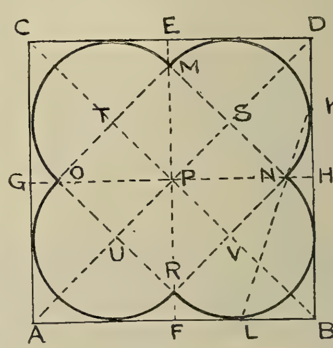


Fig. 137.

Problem 83.

Within a given circle to inscribe any number of equal semicircles having adjacent diameters. For example, *seven.*

THE HEPTAFOIL.

Find centre C (Prob. 33). Divide the circle into as many equal parts as the number of semicircles required (in this case seven), and from each of these points, A, B, D, E, F, G, and H, draw diameters. From the point A draw the tangent AM (Prob. 84), and bisect the angle CAM by a line cutting CK in N (Prob. 13). With C as centre, and CN as radius, mark off the points O, P, R, S, T, and U. Join each of these points to form the polygon. From the centre of each side of the polygon, with a radius equal to half of one of its sides, draw the seven semicircles required.

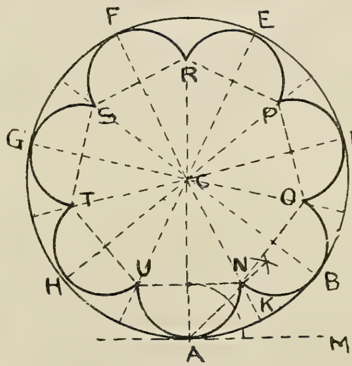


Fig. 138.

GOTHIC TRACERY.

These examples of Gothic tracery should be carefully copied to an enlarged scale. The scale should be varied: for instance, a simple figure like Fig. 141 should be enlarged in the proportion of say 5 : 4; while a more complicated figure, as Fig. 147, should be drawn twice the size of the design shown.

Fig. 139 shows the preliminary work necessary for drawing Fig. 140, and Fig. 145 the construction for Fig. 146.

After copying these examples, the student should take some of the preceding problems of *foiled figures,* and build up tracery upon them.

Gothic tracery forms an excellent example of the applica-

tion of geometrical problems to the principles of design; it also forms the best practice possible for the use of drawing instruments.

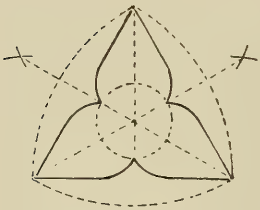


Fig. 139.

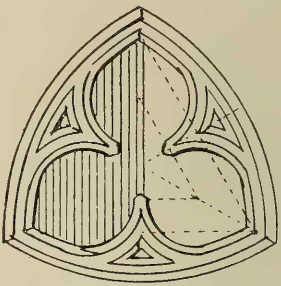


Fig. 140.

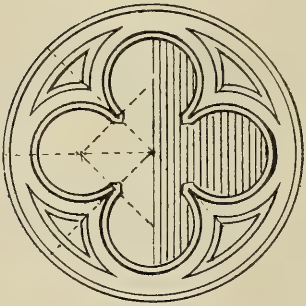


Fig. 141.

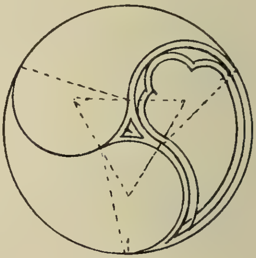


Fig. 142.

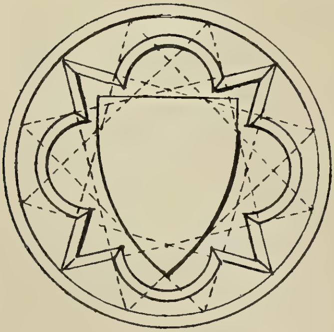


Fig. 143.

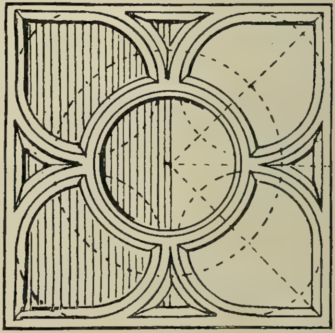


Fig. 144.

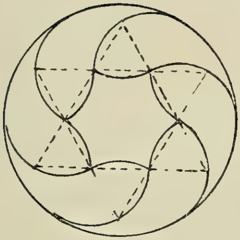


Fig. 145.

Fig. 146.

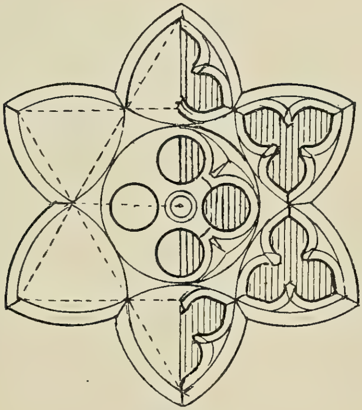


Fig. 147.

CHAPTER VI

TANGENTS AND TANGENTIAL ARCS

Problem 84.

To draw a tangent to a given circle at a given point A.

Find the centre of the circle C (Prob. 33). Join AC. From A draw the line AB perpendicular to AC, and produce it. Then AB is the tangent required.

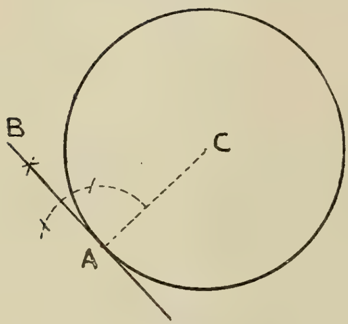


Fig. 148.

Problem 85.

To draw a tangent to a given circle from a given point A *outside it.*

Find the centre C (Prob. 33). Join AC, and bisect it in B. From B as centre, and with BA as radius, describe a semicircle cutting the circumference in D. Join AD, and produce it. Then the line AD is the tangent required.

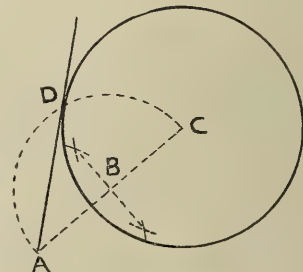


Fig. 149.

Problem 86.

To draw a tangent to the arc of a circle at a given point A *without using the centre.*

Draw the chord AB and bisect it in C (Prob. 1). At C

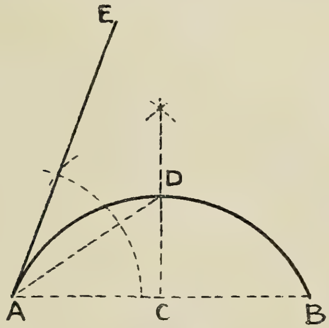


Fig. 150.

erect perpendicular to AB cutting the arc in D. Join AD. Make the angle DAE equal to DAC. Then EA produced is the tangent required.

Problem 87.

To draw a given circle with radius equal to line D *to touch two straight lines forming a given angle* ABC.

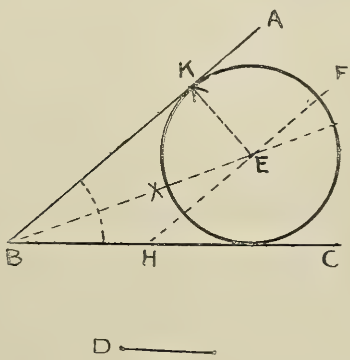


Fig. 151.

Bisect the angle ABC by the line BE. Draw the line FH

parallel to BA (Prob. 4), and at a distance equal to given line D from it. Where FH intersects BE will be the centre of the circle. From E draw the line EK perpendicular to BA. With E as centre, and EK as radius, draw the circle required.

Problem 88.

To draw tangents to a circle from a given point A, *the centre of the circle not being known.*

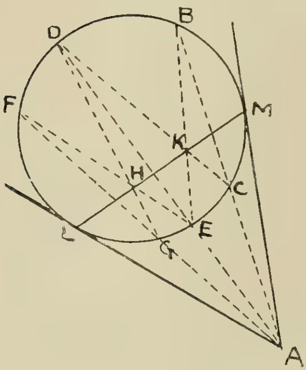


Fig. 152.

From point A draw any three secants to the circle, as CB, DE, and GF. Join BE and DC, DG and FE, by lines intersecting in the points H and K. Draw a line through H and K till it meets the circumference in L and M. Join AL and AM, which will be the tangents required.

Problem 89.

In a given angle CAB, *to inscribe a circle which shall pass through a given point* D.

Bisect the angle CAB by the line AE (Prob. 13). Take any convenient point F in AE, and from F draw the line FG perpendicular to AC (Prob. 5). With F as centre, and FG as radius, draw a circle. Join DA, cutting the circle in H. From the given point D draw DK parallel to HF. With K as centre, and KD as radius, draw the required circle.

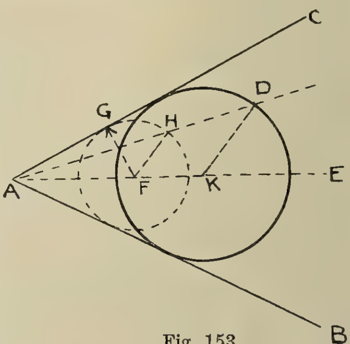


Fig. 153.

Problem 90.

To draw a circle which shall pass through the given point A *and touch a given line* BC *in* D.

At the given point D erect the line DE perpendicular to

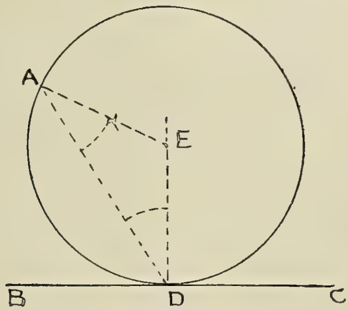


Fig. 154.

BC (Prob. 5), and join AD. At point A construct an angle DAE equal to the angle ADE (Prob. 12). AE intersects DE in E. With E as centre, and ED as radius, draw the required circle.

Problem 91.

To draw a circle which shall pass through the two given points A *and* B *and touch the given line* CD.

Join the two given points AB, and produce the line till it meets the given line CD produced in E. On AE describe the semicircle EFA; at B draw BF perpendicular to AE. From E, along the line ED, set off EG equal to EF. Then G, B, A are three points in the required circle, which can be drawn as required (Prob. 34).

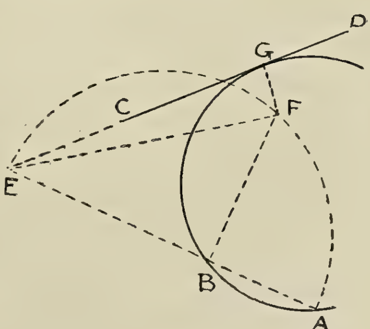


Fig. 155.

Problem 92.

To draw four equal circles, with radius equal to given line E, *to touch two given lines* AB *and* CD, *which are not parallel.*

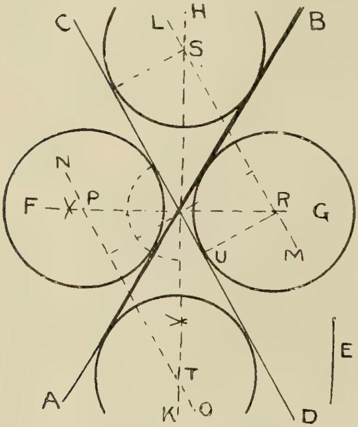


Fig. 156.

Bisect any two adjacent angles by the lines FG and HK. Draw the lines LM and NO parallel to the given line CD, at a distance from it equal to the given radius E (Prob. 4). Where these lines intersect the bisectors, we get the points S, R, T, and P. With the points R, S, T, and P as centres, and with a radius equal to E, draw the four circles required.

Problem 93.

To draw an inscribed and an escribed circle, tangential to three given straight lines, forming a triangle ABC.

Note.—An *escribed circle* is also called an *excircle.*

Bisect the angles BAC and ACB by lines intersecting in F. From F draw the line FH perpendicular to AD. With F as centre, and FH as radius, draw the inscribed circle required. Bisect the angle BCD by a line cutting the line AG in E. Draw the line EK perpendicular to AL. With E as centre, and EK as radius, draw the escribed circle required.

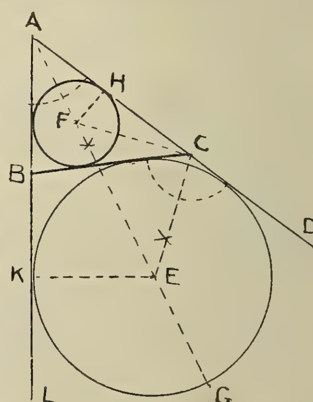


Fig. 157.

Problem 94.

A principle of inscribed and escribed circles.

If a triangle ABC be taken, and AF, BD, and CE, lines drawn from the three angles perpendicular to the opposite sides, they will all intersect at the point H. Join the points D,E,F. This will form a triangle of which H is the "incentre," being the centre of the inscribed circle. The centres of the escribed circles will be the points A, B, and C. The radii of the circles are found by drawing lines from the centres perpendicular to the sides of the triangle produced (Prob. 6), as the line BK.

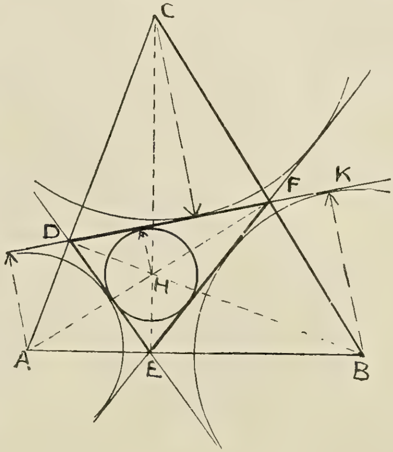


Fig. 158.

Problem 95.

To draw two circles tangential to three given straight lines, two of which are parallel.

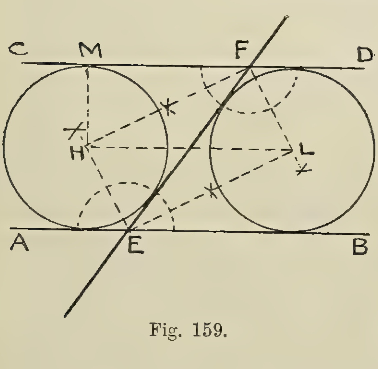


Fig. 159.

Let AB and CD be the two given parallel lines, and let the third line intersect them in E and F. Bisect the four angles AEF, BEF, CFE, and DFE by lines meeting at H and L. From H draw the line HM perpendicular to CD (Prob. 6). With H and L as centres, and a radius equal to HM, draw the two required circles.

Note.—A line joining H and L will be parallel to the two given lines AB and CD.

Problem 96.

To draw two circles tangential to three given straight lines, none of which are parallel; the third line to be drawn to cut the other two.

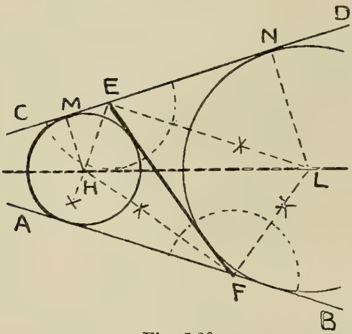


Fig. 160.

Let AB, CD, and EF be the three given lines. Bisect the four angles AFE, BFE, CEF, and DEF by lines meeting at H and L. From H and L draw the lines HM and LN perpendicular to CD (Prob. 6). With H as centre, and radius HM, draw a circle; with L as centre, and LN as radius, draw the other circle required.

Note.—A line produced through the points H and L would be the bisector of the angle formed by producing the lines AB and CD.

Problem 97.

To draw DIRECT COMMON TANGENTS *to two given circles of unequal radii.*

Let AC be the radius of one circle, and BD of the other. Join the centres A and B. From the centre A draw a circle with a radius $=$ AC $-$ BD. Bisect the line AB at E (Prob. 1). From E, with radius EA, draw a circle cutting the circle FKG in the points F and G. Join FB and GB. From F and B draw the lines FO and BR perpendicular to FB (Prob. 6); and from the points G and B draw the lines GP and BS perpendicular to GB

Draw the line HL through the points O and R, and the line MN through the points P and S; these will be the tangents required.

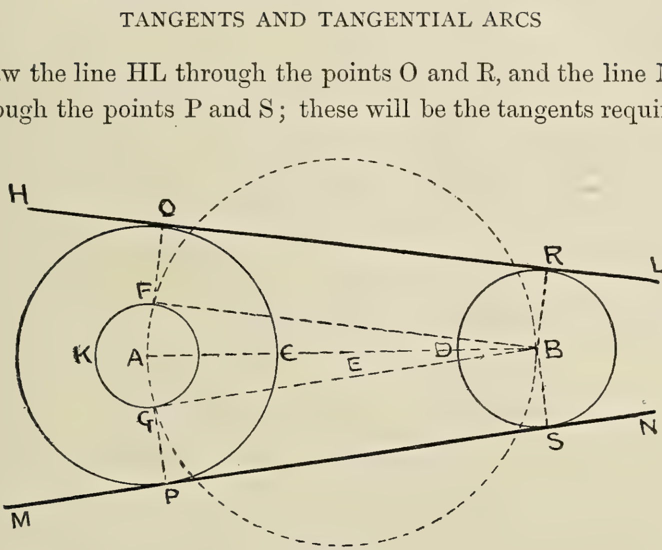


Fig. 161.

Problem 98.

To draw TRANSVERSE COMMON TANGENTS *to two given circles of unequal radii.*

Let A and B be the centres of the given circles, and AC and

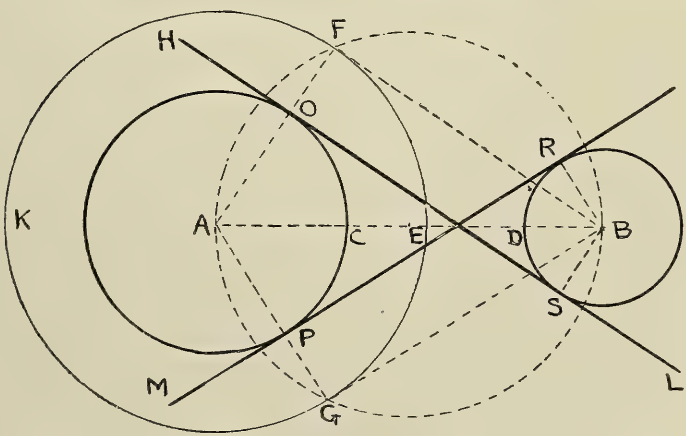


Fig. 162.

BD their radii. Join AB. With A as centre, and a radius = AC

+ BD, draw a circle. Bisect the line AB in E (Prob. 1). With E as centre, and a radius equal to EA, draw a circle cutting the circle FKG in the points F and G. Join AF and AG, cutting the given circle PCO in O and P. Join FB and GB. From B draw the line BS perpendicular to FB (Prob. 6), also the line BR perpendicular to GB. Draw the line HL through the points O and S, and the line MN through the points P and R; these will be the tangents required.

TANGENTIAL CIRCLES AND ARCS.

Problem 99.

Showing the principle of tangential circles.

One circle can touch another circle either internally or externally, and any number of circles can be drawn to touch a given line, as well as each other, in the same point. For instance take the point C on the given line AB. All circles that touch a given line in the same point touch each other at that point; and all their centres will be on a line perpendicular to the given line.

If they are on the opposite sides of the given line, they will touch externally; and if on the same side, will touch internally. If they are on the same side of the line, one circle will be entirely within the other.

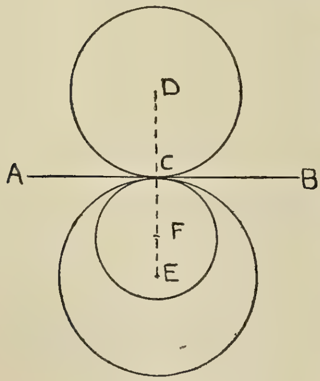


Fig. 163.

If their centres F and E are on the same side of the given line AB, the distance between them is equal to the difference of the radii; but if their centres E and D are on opposite sides of the given line, the distance between them is equal to the sum of their radii.

The point of contact can always be found by joining their centres.

Problem 100.

To draw four equal circles with radius equal to given line D, *with their centres on a given line* AB; *two to touch externally and two internally a given circle, whose centre is* C *and radius* CG.

With centre C, and radius equal to the sum of the radii, *i.e.* CG + D, draw a circle cutting the given line AB in H and N. With centre C, and radius equal to the difference of the radii, *i.e.* CG − D, draw a circle cutting the given line AB in L and M. With H, L, M, and N as centres, and radius equal to D, draw the four circles required.

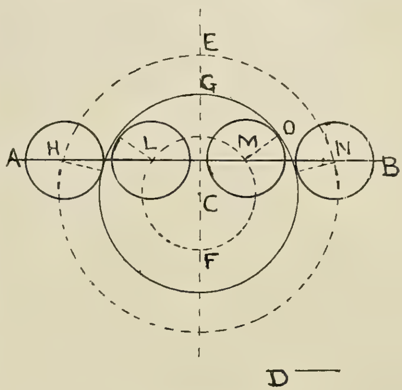


Fig. 164.

Problem 101.

To draw four equal circles, with radius equal to given line D, *with their centres on a given arc* AB; *two to touch externally and two internally a given circle, whose centre is* C *and radius* CG.

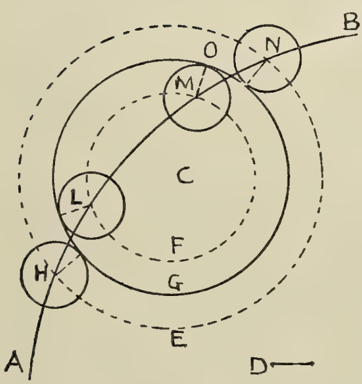


Fig. 165.

The construction of this problem answers word for word the same as the last, the only difference being the words *given arc* instead of *given line.*

Problem 102.

To describe a circle tangential to and including two given equal circles A *and* B, *and touching one of them in a given point* C.

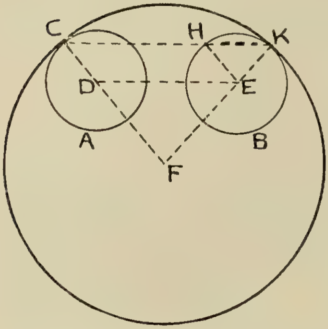


Fig. 166.

Find the centres of the two given circles D and E, and join them (Prob. 33). Join CD. From C draw the line CK parallel to DE, meeting the given circle B in K. Join KE, and produce it till it meets CD produced in F. With F as centre, and radius FC, draw the required circle.

Problem 103.

To describe a circle tangential to and including two unequal given circles A *and* B, *and touching one of them in a given point* C.

Find the centres D and E. Join CE. Cut off from CE, CH equal in length to the radius of the smaller circle. Join DH. Produce CE. At D construct the angle HDF equal to the angle DHF (Prob. 12). DF meets the line CE produced in F. With F as centre, and FC as radius, draw the circle required.

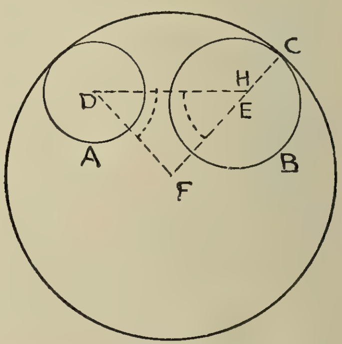


Fig 167.

PROBLEM 104.

To draw the arc of a circle having a radius of $1\frac{1}{4}$ inches, which shall be tangential to two given unequal circles A *and* B *and include them*

Note.—The diameters and distance between the circles must not be greater than $2\frac{1}{2}$ inches.

Find the centres D and E (Prob. 33), and produce a line through them indefinitely cutting the circles in K and L in both directions. From the points K and L on this line, set off KF and LH equal to the radius of the required arc, viz. $1\frac{1}{4}$ inches. With D as centre, and a radius equal to DF, draw an arc at M ; and with E as centre, and EH as radius, draw an arc intersecting the other arc at M. From M draw the line MD, and produce it till it meets the circumference of the larger circle in C. With M as centre, and MC as radius, draw the required arc.

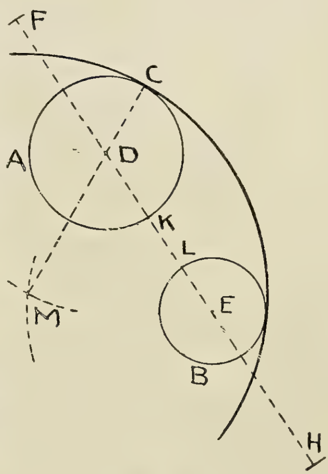


Fig. 168.

PROBLEM 105.

To inscribe in a segment of a circle, whose centre is E, *two given equal circles with a radius equal to line* D.

From any radius EF cut off FL = D, and describe a circle with radius EL. Draw the line KL parallel to the base of the segment, and at a distance equal to given radius

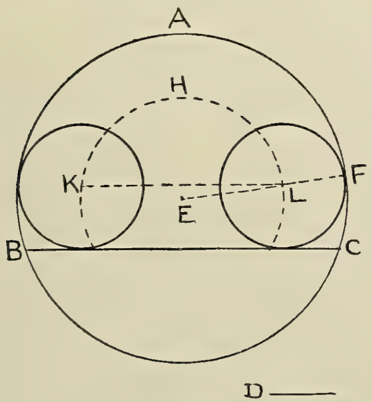


Fig. 169.

D from it (Prob. 4). Join EL and produce it till it meets the circumference in F. With L and K as centres, and LF as radius, draw the two required circles.

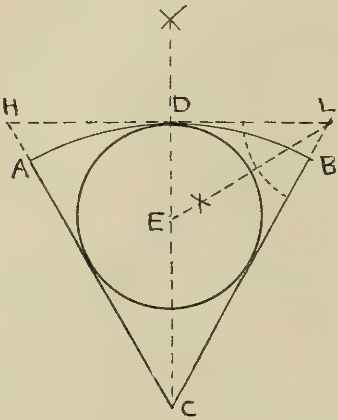


Fig. 170.

Problem 106.

In the given sector of a circle ABC, *to inscribe a circle tangential to it.*

Bisect the angle ACB by the line D (Prob. 13). At D draw a tangent HL (Prob. 84) to meet the sides of the sector produced. Bisect the angle CLH by a line cutting CD in E. With E as centre, and ED as radius, draw the circle required.

Problem 107.

Draw a circle having a radius of $\frac{3}{8}$ of an inch tangential to two given unequal circles A *and* B *externally.*

Note.—The circles must not be more than $\frac{3}{4}$ inches apart.

Find the centres D and E of the given circles (Prob. 33). From centre D, with the sum of the radii, *i.e.* DK + $\frac{3}{8}$ of an inch, draw an arc at H; and from centre E, with the sum of the other radii, *i.e.* EL + $\frac{3}{8}$ of an inch, draw another arc at H. With H as centre, and radius HK, draw the circle required.

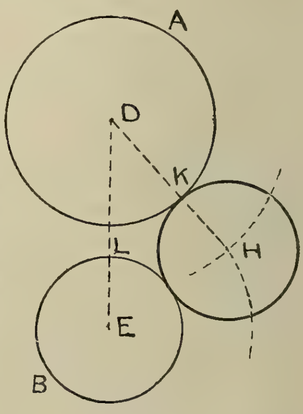


Fig. 171.

Problem 108.

To draw the arc of a circle tangential to two given unequal circles A *and* B *externally, and touching one of them in a given point* C.

Find the centres D and E of the given circles (Prob. 33). Join CE, and produce it indefinitely. Set off from C, on CE produced, CH equal to the radius of the larger given circle. Join DH. At D construct an angle HDF equal to the angle FHD, to meet EC produced in F. With F as centre, and radius FC, draw the arc required.

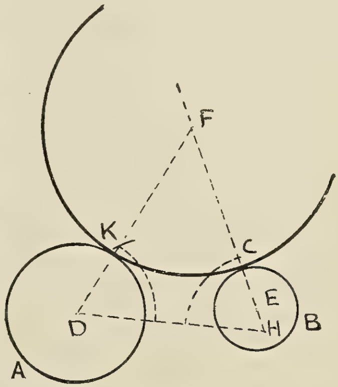


Fig. 172.

Problem 109.

To draw a circle, with a radius equal to given line C, *tangential to two given unequal circles* A *and* B, *to touch* A *externally and* B *internally.*

Note.—The given radius must be greater than half the diameter of the enclosed circle and the distance between the circles.

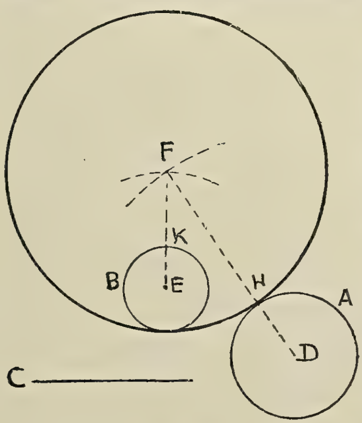


Fig. 173.

Find the centres D and E of the two given circles (Prob. 33). From centre D, with the sum-radius, *i.e.* DH + C, describe an arc at F; and with E as centre, and the difference-radius, *i.e.* C — EK, draw another arc cutting the other at point F. Join FD and FE. With F as centre, and FH as radius, draw the circle required.

Problem 110.

To draw a circle of $\frac{3}{8}$ of an inch radius tangential to the given line AB *and the given circle* CDE.

Note.—The circle must be less than $\frac{3}{4}$ inches from the line.

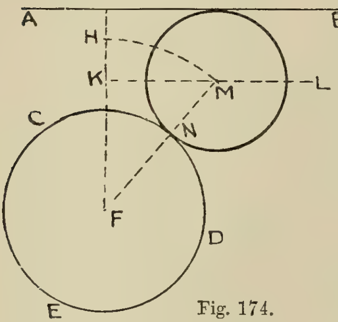


Fig. 174.

Draw the line KL parallel to AB, and $\frac{3}{8}$ of an inch from it. Find the centre F of the given circle (Prob. 33), and with the sum-radius of the two circles draw the arc HM cutting KL in M. Join FM. With M as centre, and MN as radius, draw the required circle.

Problem 111.

To draw a series of circles tangential to each other, and to two lines converging towards an inaccessible point.

Let AB and CD be the two converging lines. To draw the

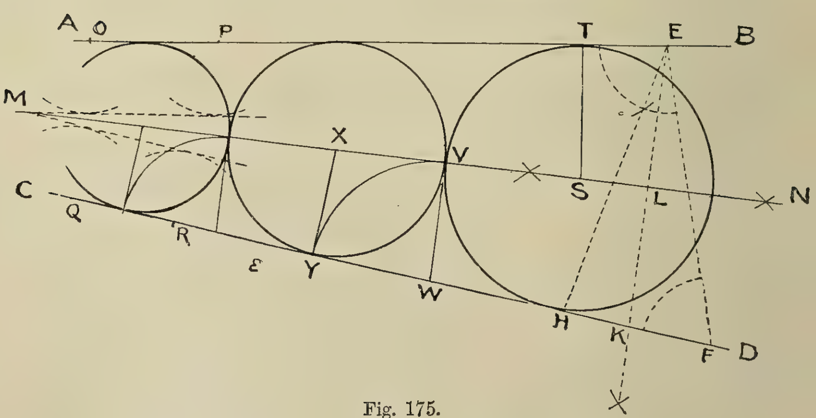


Fig. 175.

bisector MN, first draw any line EF, and at E construct the angle AEH equal to the angle CFE; then bisect the angle HEF by the line EK, and bisect EK in L. Draw a line through L perpendicular to EK, which will be the bisector. The bisector

could also be drawn by finding a point midway between the converging lines at each end, in the following manner. Take any radius smaller than one-half the distance between the two lines, and with it draw four arcs from any points O, P, Q, and R, two on each line. Draw tangents to these arcs till they meet each other in M, which will be a point midway between the converging lines. Find a similar point at the other end, and join them, which will give the bisector.

From any point S draw the line ST perpendicular to AB. With S as centre, and ST as radius, draw a circle. From the point V draw the line VW perpendicular to MN. With W as centre, and radius WV, draw the arc VY. From Y draw the line YX perpendicular to CD. With X as centre, and radius XY, draw another circle. Repeat the same construction as many times as required.

APPLICATION OF TANGENTIAL ARCS TO DESIGN.

If two circles, or their arcs, meet each other on a line joining their centres, they must be tangential to each other.

Fig. 176 is an illustration of this rule. The centres of several circles are shown by the small circles, and are connected to their respective arcs by dotted lines and arrow-heads.

Figs. 177, 178, and 182 are applications of this principle to design.

These designs should be copied to a larger scale, after which the student should attempt to design other simple forms on the same principle. This will not only tend to develop the inventive faculty, but will impress the principle on the mind.

Fig. 179 is an example of a *Trochilus* or *Scotia:* it usually forms part of the base of a column. When it is of small size, it is generally drawn by tangential arcs touching internally, as the arc from *a* to *b* (Fig. 176); but when the curve forms part of a large object, the pedestal of a statue for instance, it has a better appearance if constructed as shown.

Figs. 180 and 181 are examples of continuous ornament.

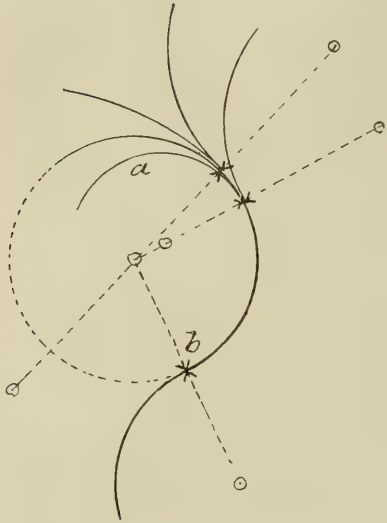


Fig. 176.

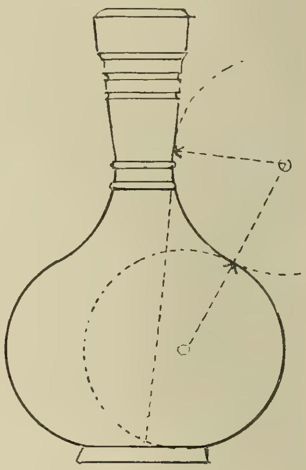


Fig. 177.

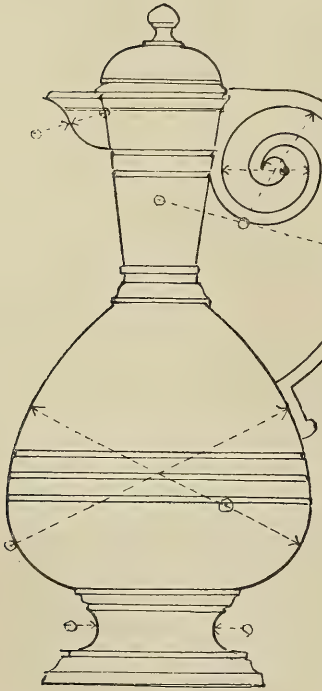


Fig. 178.

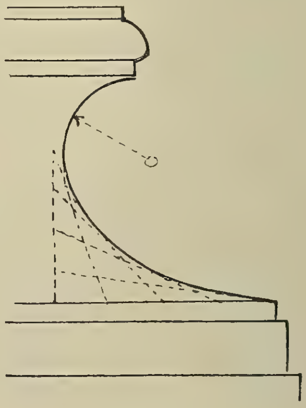


Fig. 179.

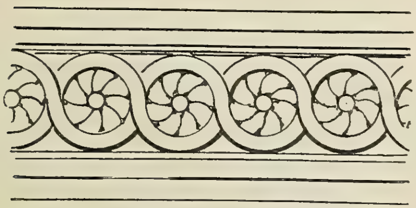


Fig. 180.

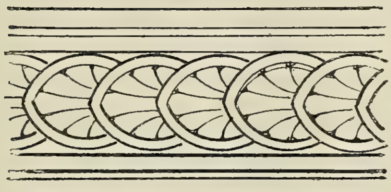


Fig. 181.

Fig. 182.

Fig. 183.

Fig. 180 is called a *Guilloche:* it would afford good practice to the student to draw this between two concentric arcs instead of parallel lines.

Fig. 183 shows how different geometrical figures may be combined to form geometrical patterns.

CHAPTER VII

PROPORTIONAL LINES

Proportional lines may be illustrated by the example of *simple proportion* in arithmetic, in which we have four terms, *e.g.*—
$2 : 4 : : 5 : 10$.

The *relationship* or *ratio* between the first two terms with regard to magnitude is the same as that between the second two, *e.g.* as 2 is to 4, so is 5 to 10 ; therefore these four numbers are said to be in proportion.

The first and fourth terms are called the *extremes*, and the second and third the *means*.

The product of the extremes equals the product of the means, *e.g.* $2 \times 10 = 4 \times 5$. So, the first three terms being given, we can find the fourth. If we divide the product of the means by the first extreme we get the *fourth proportional, e.g.* $\frac{4 \times 5}{2} = 10$.

Almost all geometrical questions on proportion are based on the following theorems :—

Take any triangle ABC, and draw any line DE parallel to one side, then—

CD : DA : : CE : EB
CD : CE : : CA : CB
CE : ED : : CB : BA
CD : DE : : CA : AB.

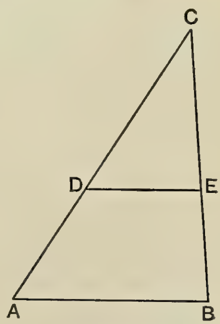


Fig. 184.

There are five varieties of proportional lines, viz.—

Greater fourth proportional.

Less fourth proportional.
Greater third proportional.
Less third proportional.
Mean proportional.

If the quantities be so arranged that the second term is greater than the first,—as $4 : 6 :: 8 : x$,—the last term is called the *greater fourth proportional.*

If the terms are arranged so that the second term is less than the one preceding,—as $8 : 6 :: 4 : x$,—the last or unknown term is called the *less fourth proportional.*

When the two means are represented by the same number, —thus $4 : 6 :: 6 : x$,—the answer or x is called the *third proportional.*

The third proportional is found by dividing the square of the second by the first, *e.g.*—

$$\frac{6^2}{4} \text{ or } \frac{6 \times 6}{4} = 9.$$

If the terms are placed so that the larger number is repeated,—thus $4 : 6 :: 6 : x$,—the last term is called the *greater third proportional;* but if the terms are arranged so that the smaller number is repeated,—as $6 : 4 :: 4 : x$,—the result is called the *less third proportional.*

The mean proportional between any two numbers is found by extracting the square root of their product,—*e.g.* $4 \times 9 = 36$; the square root of $36 = 6$, which is the *mean proportional.*

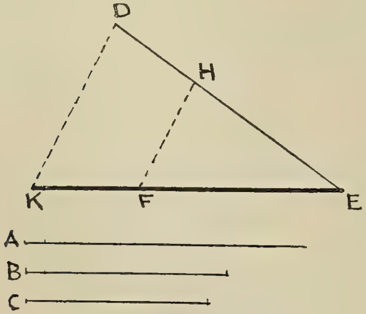


Fig. 185.

PROBLEM 112.

To find a fourth proportional to three given lines A, B, *and* C. THE GREATER FOURTH PROPORTIONAL.

Draw EH equal to given line C, and EF, at any angle with it, equal to given line B. Join FH, and produce EH to

D, making ED equal to given line A. Draw DK parallel to FH till it meets EF produced in K (Prob. 4). Then KE will be the *greater fourth proportional* to the lines A, B, and C, *i.e.* C : B : : A : KE, *e.g.* if C = 6 feet, B = 8 feet, A = 12 feet, then KE = 16 feet.

Problem 113.

To find a fourth proportional to three given lines A, B, and C. THE LESS FOURTH PROPORTIONAL.

Draw the line DE equal to given line A, and EF, at any angle with it, equal to given line B. Join FD. From E, along ED, set off EG equal to given line C. From G draw GH parallel to FD (Prob. 4). Then HE is the *less fourth proportional* to the given lines A, B, and C, *i.e.* A : B : : C : HE.

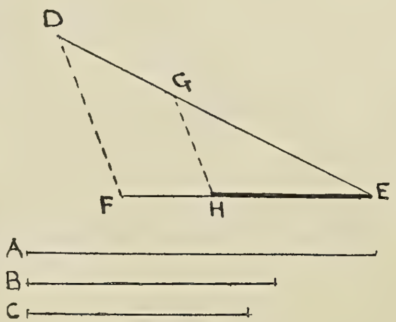


Fig. 186.

Problem 114.

To find a third proportional between two given lines A *and* B. THE GREATER THIRD PROPORTIONAL.

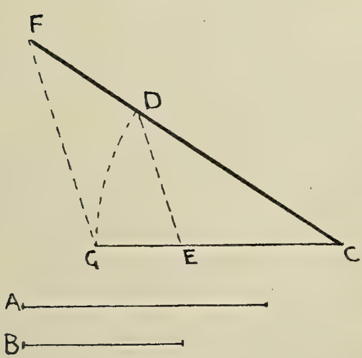


Fig. 187.

Draw CD equal to given line A, and CE, at any angle with it, equal to given line B. Join DE. With C as centre, and CD as radius, draw the arc DG to meet CE produced in G. From G draw the line

GF parallel to DE till it meets CD produced in F (Prob. 4). Then CF is the *greater third proportional* to the given lines A and B, *i.e.* B : A : : A : CF.

Problem 115.

To find a third proportional between two given lines A *and* B.

THE LESS THIRD PROPORTIONAL.

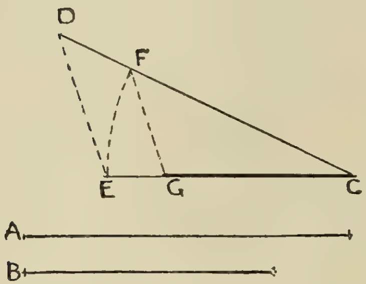


Fig. 188.

Draw CD equal to the given line A, and CE, at any angle with it, equal to given line B. Join DE. From C as centre, and with radius CE, draw the arc EF cutting CD in F. Draw FG parallel to DE (Prob. 4). Then CG is the *less third proportional* to the given lines A and B, *i.e.* A : B : : B : CG.

Problem 116.

To find the MEAN PROPORTIONAL *between two given lines* AB *and* CD. See also the description to Fig. 222.

Produce the given line AB to E, making AE equal to the given line CD. Bisect the line EB in H (Prob. 1). From H as centre, and with radius HB, draw the semicircle EKB. At A draw the line AK perpendicular to EB, cutting the semicircle in K (Prob. 5). Then AK is the *mean proportional* to the given lines AB and CD, *e.g.* if AB = 9 feet and CD = 4 feet, then AK = 6 feet.

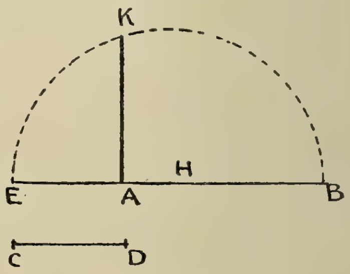


Fig. 189.

PROBLEM 117.

To divide a line in MEDIAL SECTION, *i.e. into* EXTREME *and* MEAN *proportion.*

Let AB be the straight line. At A draw AC perpendicular to AB, and equal to it (Prob. 5). Bisect AC in D (Prob. 1). With D as centre, and DB as radius, draw the arc cutting CA produced in E. With A as centre, and AE as radius, draw the arc cutting AB in F. Then AB : AF : : AF : BF.

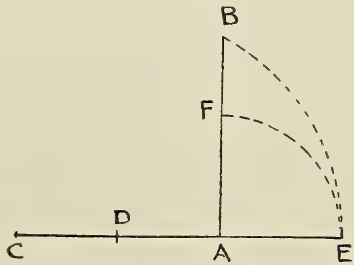


Fig. 190.

PROBLEM 118.

To divide any straight line AB *in the point* C, *so that* AC : CB : : 3 : 4.

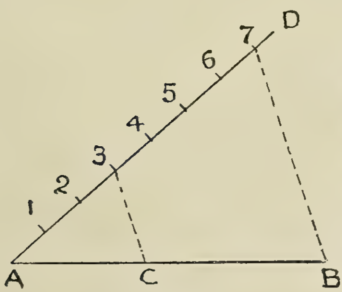


Fig. 191.

At A draw the line AD of indefinite length, and at any angle to AB. From A, along AD, mark off seven equal distances of any convenient length. Join 7B. At 3 draw the line 3C parallel to 7B (Prob. 4). Then AC : CB : : 3 : 4, or, AB : AC : : 7 : 3.

PROBLEM 119.

To construct a triangle on a given line AB, *so that the three angles may be in the proportion of* 2 : 3 : 4.

From B, with any radius, describe a semicircle and divide it into nine equal parts (Prob. 15). Draw the lines B4 and B7.

Then the three angles 7BC, 4B7, and 4BA are in the proportion of 2 : 3 : 4. The sum of the three angles are also equal to two right angles, because a semicircle contains 180°. They must also be the three angles of a triangle, because the three angles of any triangle are together equal to two right angles. From A draw the line AD parallel to B7 till it meets B4 produced in D (Prob. 4). Then ABD is the triangle required.

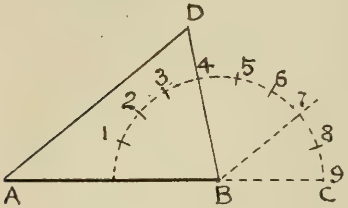


Fig. 192.

PROBLEM 120.

This problem illustrates an important principle in proportion.

Take a triangle ABC, the sides of which shall bear a certain

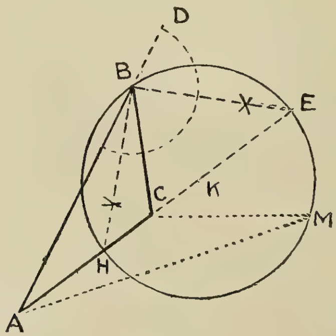


Fig. 193.

ratio. For example, let AB : BC as 2 : 1. Produce AB to D, and bisect the angles ABC and DBC by lines meeting AC produced in H and E. Bisect the line HE in K (Prob. 1). With K as centre, and KE as radius, draw the circle EBH. Now, if we take any point M in this circle, and join MA and MC, we

shall find that they bear the same ratio as the lines AB and BC. In the example given MA : MC as 2 : 1. The same result would be obtained from any point in the circle.

Problem 121.

To divide a right angle into five equal parts.

Let ABC be the right angle. Divide BC in D, so that BC : BD :: BD : DC (Prob. 116). With C as centre, and CB as radius, describe the arc BE; and with B as centre, and BD as radius, describe the quadrant DF, cutting BE in E. FE is one-fifth of the quadrant FD. Arcs equal to it set off on FD will divide it into five equal parts.

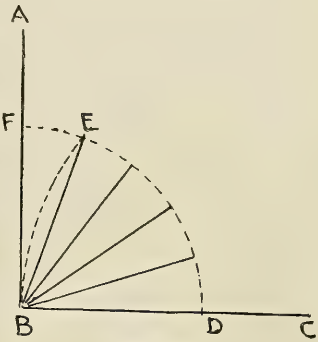


Fig. 194.

Problem 122.

To find the Arithmetic, the Geometric, and the Harmonic means between two given lines AB *and* BC.

Bisect AC in D (Prob. 1). With D as centre, and radius DA, draw the semicircle AEC. At B draw BE perpendicular to AC (Prob. 5). Join DE. From B draw BF perpendicular to DE.

AD is the Arithmetic, BE the Geometric, and EF the Harmonic mean between the two lines as required.

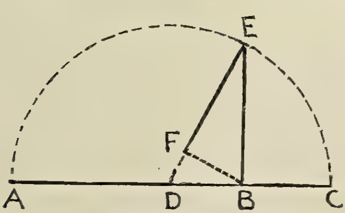


Fig. 195.

PROBLEM 123.

Taking the given line AB, *one inch long, as the unit; find lines representing* $\sqrt{2}$ *and* $\sqrt{3}$

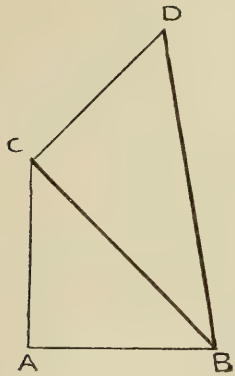


Fig. 196.

Draw AC perpendicular to AB, and of the same length (Prob. 7). Join CB. $CB = \sqrt{2}$.

Draw CD perpendicular to CB, and one inch in length. Join DB. $DB = \sqrt{3}$.

If AB is the edge of a cube, CB = the diagonal of its face, and DB the diagonal of the cube, which are therefore to one another as $1 : \sqrt{2} : \sqrt{3}$.

CHAPTER VIII

EQUIVALENT AREAS

It is advisable that the student should thoroughly master the following illustrations and explanations on the relation between the various kinds of *triangles* and *quadrilaterals* and their several equivalent areas, as they form the basis of all problems on equivalent areas in rectilinear figures.

Triangles with equal bases and equal altitudes are also equal in area.

Triangles standing on the same base and drawn between parallel lines are equal in area, e.g.—

The triangle A is equal to the triangle B (Fig. 197). A is an

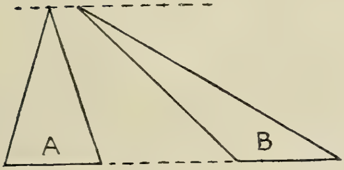


Fig. 197.

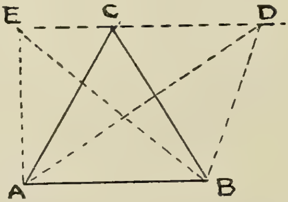


Fig. 198.

isosceles triangle, and can be converted into the scalene triangle B; or B could be converted into the isosceles triangle A.

An equilateral triangle ABC (Fig. 198) could be converted into the scalene triangle ABD, or either could be transformed into the right-angled triangle ABE.

Precisely the same principle applies to *parallelograms, e.g.*—

Parallelograms with equal bases and equal altitudes are also equal in area.

The rectangle A is equal in area to the rhomboid B (Fig. 199).

The square ABDC (Fig. 200) can be converted into the

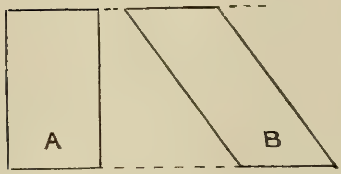


Fig. 199.

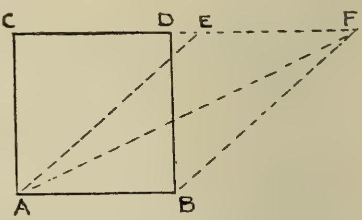


Fig. 200.

rhomboid ABFE, or the rhomboid can be converted into an equivalent square.

If triangles and parallelograms stand upon the same or equal bases, and between the same parallels, the triangle is half the area of the parallelogram, e.g.—

The triangle ABF (Fig. 200) is half of the rhomboid ABFE, as well as half of the square ABDC.

A triangle standing on double the base of the parallelogram, and the same altitude, is equal in area to the parallelogram.

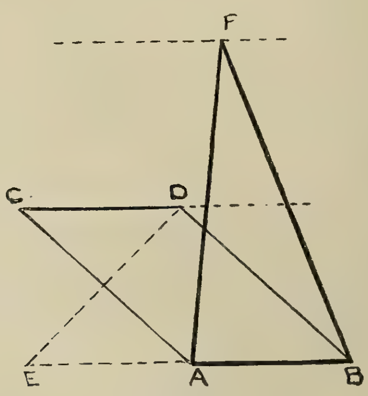


Fig. 201.

The triangle EBD (Fig. 201) is equivalent to the rhomboid ABCD.

A triangle standing on the same, or equal base as the parallelogram, and double its vertical height, is equal in area to the parallelogram. The triangle ABF (Fig. 201) is equivalent to the rhomboid ABCD, as well as to the triangle EBD; so each of these figures can be converted into the other two.

Problem 124.

On a given line AB *to construct an isosceles triangle equal in area to a given triangle* ABC.

Bisect the base AB in E, and erect the line ED perpendicular to AB (Prob. 1). From point C draw the line CD parallel to AB (Prob. 4). Draw the lines DA and DB. Then ABD is the triangle required.

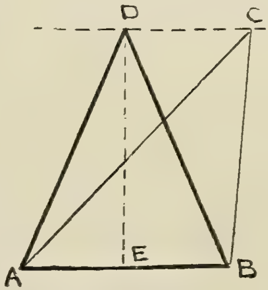


Fig. 202.

Problem 125.

To construct a rhomboid equal in area to a given triangle ABC, *and having one of its angles equal to given angle* D.

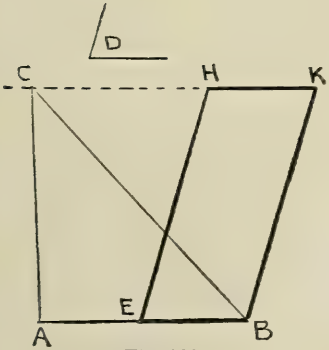


Fig. 203.

Bisect the line AB in E (Prob. 1). From point C draw the line CK parallel to AB (Prob. 4). At E construct the angle BEH equal to the given angle D (Prob. 12). From H set off HK equal in length to EB. Join BK. Then EBKH is the rhomboid required.

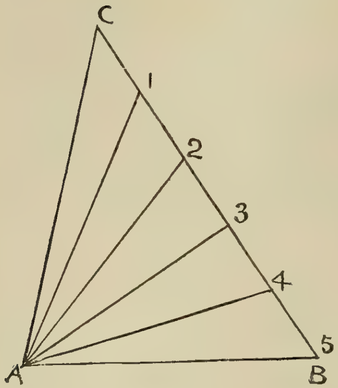


Fig. 204.

Problem 126.

To divide a given triangle ABC *into any number of equal parts,* for example five.

Divide any side of the triangle into five equal parts (Prob. 10), and draw lines to the opposite angle. We have now five triangles, and as they have equal bases and equal altitudes, they must all be equal in area to each other.

Problem 127.

To transform a given trapezium ABDC *into an equivalent triangle.*

Join BC. From D draw the line DE parallel to CB till it meets AB produced in E (Prob. 4). Join CE. Then ACE will be the triangle required.

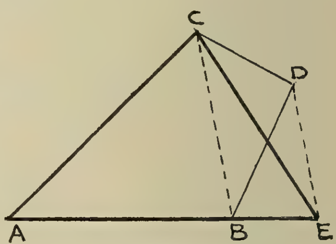


Fig. 205.

We have simply converted the triangle BCD, which forms one-half of the trapezium BCDE, into the triangle BCE; and as the two triangles have the same base, and are drawn between parallel lines, they must be equal to each other.

Problem 128.

To convert a given regular pentagon ABEDC *into a trapezium, or to construct an isosceles triangle of an area equal to that of a given pentagon.*

Join A and B to D. From point E draw the line EH

parallel to DB till it meets AB produced in H (Prob. 4). Join DH. Then ACDH is the required trapezium. From C draw the line CF parallel to DA till it meets AB produced in F. Join DF. Then FDH is the isosceles triangle required.

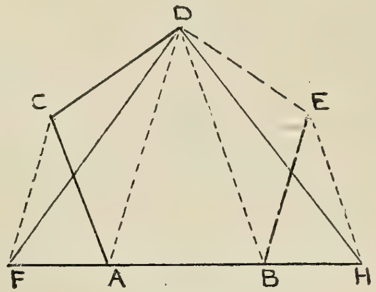


Fig. 206.

PROBLEM 129.

To draw a triangle equivalent to any rectilinear figure, for example an irregular hexagon ACDEFB.

Join DA. From C draw CH parallel to DA to meet AB produced in H (Prob. 4). Join DH.

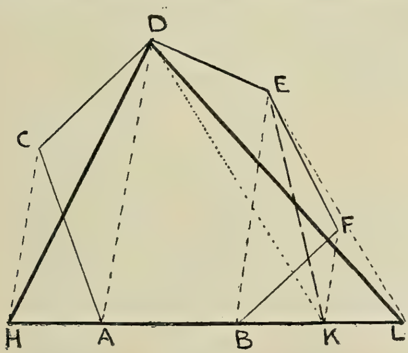


Fig. 207.

We now have an irregular pentagon HDEFB equivalent to the given hexagon.

Join EB. From F draw FK parallel to EB, to meet AB produced in K. Join EK.

We now have a trapezium HDEK equal in area to the given hexagon.

Join DK. From E draw the line EL parallel to DK till it meets AB produced in L. Join DL.

We now have the required triangle HDL equivalent to the given hexagon.

PROBLEM 130.

To construct a triangle equal in area to a given irregular polygon ACDEFGB, *the polygon having a re-entering angle* BGF.

Join CE, and at D draw the line DH parallel to it till it

meets AC produced (Prob. 4). Join EH. It is clear that the triangle CHE is equal to the triangle CDE, because they have the same base, and are between parallel lines. Join EA. From H draw HK parallel to EA. The triangle AKE must now be equal to AHE, for the same reason as just stated. Join FB, and draw the line GL parallel to it. Join FL. The triangle GLF will then be equal to the triangle GLB. We have converted the polygon so far into the trapezium KEFL. Join EL, and from F draw FM parallel to EL till it meets AB produced in M. Join EM. Then the triangle ELM is equal to the triangle ELF. EKM is the required triangle.

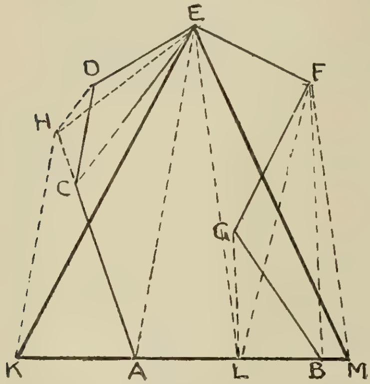


Fig. 208.

Problem 131.

To divide a given triangle ABC *into two equal parts by a line drawn from a given point D in one of its sides.*

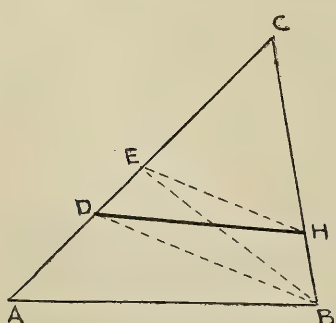


Fig. 209.

Bisect the side AC in E (Prob. 1), and join EB. The triangle is now divided into two equal parts. Join DB. From E draw the line EH parallel to DB (Prob. 4). Join DH. The triangle DBH is equal to the triangle DBE, so the line DH must bisect the given triangle ABC.

Problem 132.

To trisect a given triangle ABC *by lines drawn from a given point* D *in one of its sides.*

Trisect the side AC in the points E and F (Prob. 10). Join DB. From E draw the line EH parallel to DB (Prob. 4). Join DH. Draw the line FL parallel to EH, and join DL. Then the lines DH and DL trisect the given triangle ABC.

It will be noticed that the construction of this problem is on the same principle as the last, only the line joining EB is omitted in this instance, as it was only necessary to elucidate the construction.

A triangle can be divided into any number of equal parts on the same principle.

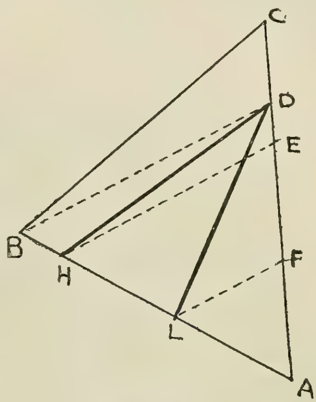


Fig. 210.

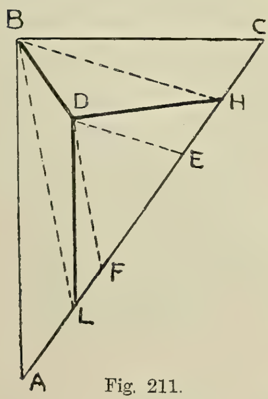


Fig. 211.

Problem 133.

To trisect a given triangle ABC *from a given point* D *inside it.*

Trisect the line AC in E and F (Prob. 10). Join DE and DF. Draw BH parallel to DE (Prob. 4), and join DH. Draw BL parallel to DF, and join DL. Join BD. Then the lines BD, DH, and DL will trisect the given triangle ABC.

Problem 134.

To bisect a parallelogram ABCD *by a line drawn from a given point* E *in one of its sides.*

Draw the diagonals AD and CB intersecting in K. From

the given point E, draw EK, and produce it to H. Then the line EH will bisect the parallelogram ACDB. The triangle AEK is equal to the triangle DHK, and the triangle BHK is equal to the triangle CEK.

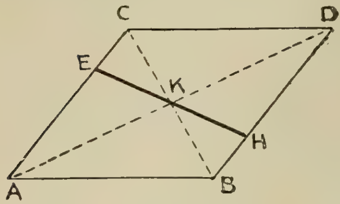


Fig. 212.

Note. — A straight line drawn from any point in any of its sides, if drawn through the centre, will bisect the parallelogram.

Problem 135.

To bisect a trapezium by a line drawn from one of its angles.

Let ACDB be the trapezium. Draw the diagonal AD and bisect it in E (Prob. 1). Join EC and EB. The trapezium is now divided into two equal parts, ACEB being one part and CEBD the other. Draw the diagonal CB, and from E draw EH parallel to CB (Prob. 4). Join CH. The triangle CBH is now equal to the triangle CBE, therefore the line CH bisects the trapezium ABCD.

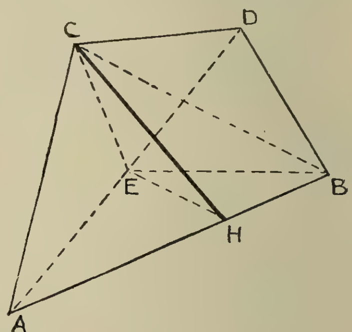


Fig. 213.

Problem 136.

To divide a given irregular polygon ABCDEFGHJK *into any number of equal areas*—for example, twelve—*by lines drawn through one angle* E.

Join each angle to E. From F draw the line FL parallel to EG till it meets HG produced in L (Prob. 4). From L draw

the line LM parallel to EH till it meets JH produced in M. From M draw the line MN parallel to EJ till it meets KJ produced in N. From N draw the line NP parallel to EK till it meets AK produced in P. We now treat the opposite side in precisely the same way, and we eventually obtain the point Q. Divide PQ into twelve equal parts, as numbered. From point

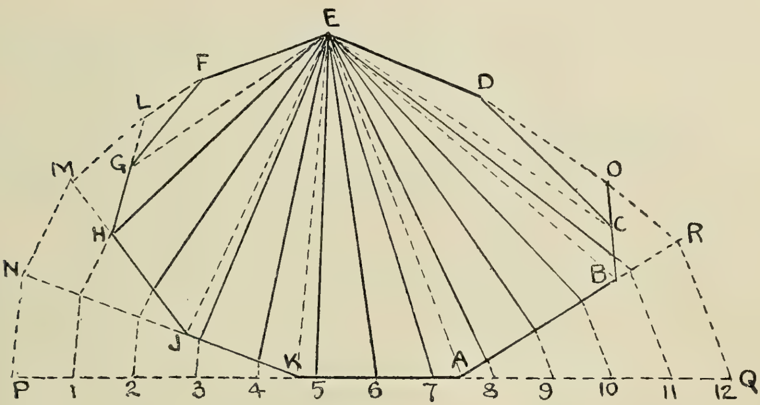


Fig. 214.

1 draw a line parallel to PN till it meets KN, then parallel to NM till it meets the given polygon, then draw it to E; the space cut off from the polygon by this line represents one of the twelve divisions required. From point 2 draw a line parallel to PN, then parallel to NM till it meets the polygon, after which join it to E; this line will cut off the second division required. Repeat this operation till the whole of the twelve required divisions are obtained.

CHAPTER IX

THE RIGHT-ANGLED TRIANGLE—PROPORTIONAL AND EQUIVALENT AREAS

Before leaving the subject of equivalent areas, it would be advisable to draw the pupil's attention to Euclid, 1st Book, Prob. 47—

In any right-angled triangle, the square which is described on the side opposite the right angle is equal to the squares described on the sides which contain the right angle; e.g.—

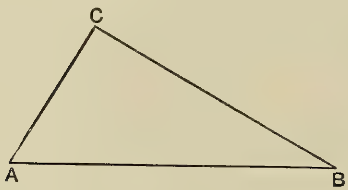


Fig. 215.

The square described on AB is equal to the sum of the squares described on AC and CB; *i.e.*—

$$AB^2 = AC^2 + CB^2.$$

Let ABC (Fig. 216) be a right-angled triangle, on which construct the three squares AFHB, ACKL, and CDEB. Find the centre of the square AFHB in the usual manner, by drawing diagonals, and through this centre draw lines parallel to CD and CB. These lines will divide the square into four equal trapezia. Bisect each side of the square CDEB, and draw a line from each point of bisection in CD and BE parallel to AB, and from each point of bisection in CB and DE parallel to AF. We shall now find that the small square a' enclosed by these lines is equal to the square a, and that the four trapezia round the square are in every way equal to the trapezia in the square AFHB; *e.g.* $b = b'$, $c = c'$, $d = d'$, and $e = e'$.

Another illustration. Let ABC (Fig. 217) be a right-angled triangle, on which construct the three squares AFHB, ACKL, and CDEB.

If we draw lines through the various angles parallel to the

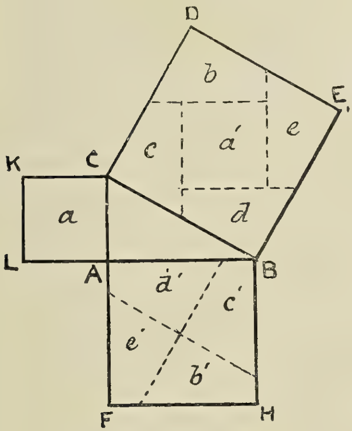


Fig. 216.

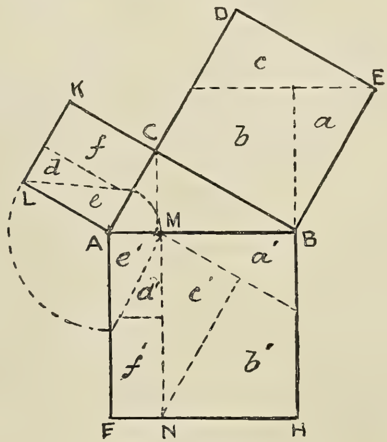


Fig. 217.

different sides, as shown by dotted lines, we are able to exhibit the truth of this proposition. The divisions that bear the same letters are equal to each other; *e.g.* $a = a'$, $b = b'$, etc. From this we can see that the rectangle AFNM is equal to the square ALKC, and the rectangle MNHB is equal to the square CDEB.

From this proposition we can prove that a triangle that has sides of 3 inches and 4 inches in length must have its hypotenuse 5 inches in length, for $3^2 + 4^2 = 5^2$.

Problem 137.

To construct a square equal in area to two given squares, the sides of which are equal to the given lines A *and* B.

At one end of the given line A erect a perpendicular equal

to the given line B (Prob. 7). Join A with the end of this line

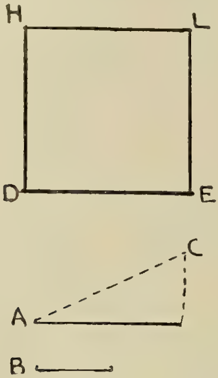


Fig. 218.

C. Draw DE equal to AC, and construct a square on DE (Prob. 26). DELH is the square required.

Problem 138.

To construct a square equal in area to the difference between two squares whose sides are equal to the given lines A *and* B.

Draw the line CD equal in length to the given line B, and at D erect the line DE perpendicular to it (Prob. 7). With C as centre, and a radius equal to the given line A, draw the arc cutting DE in E. On DE construct the square DEHL (Prob. 26), as required.

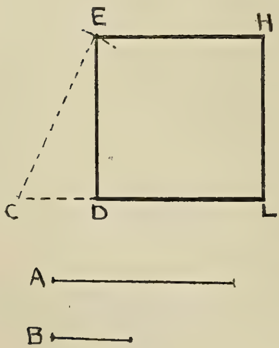


Fig. 219.

The areas of similar figures are to one another as the squares on their corresponding sides.

If we were to draw similar triangles on the lines AB, AC, and CB, the triangle on AB would equal in area the sum of the other two.

If we take AC and AB to represent the radii or diameters of two circles, then AB would represent the radius or diameter of a circle having an area equal to the sum of the other two.

If AC and CB represent the sides of similar polygons, or the radii or diameters of circles describing similar polygons, the area of the figure constructed on AB represents the sum-area of the two figures constructed on AC and CB.

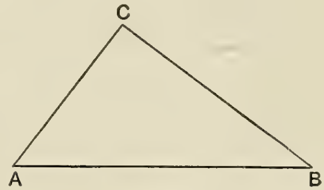


Fig. 220.

PROBLEM 139.

To construct a triangle similar to a given triangle ABC, *but of twice its area.*

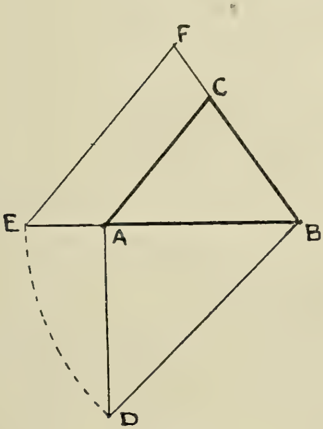


Fig. 221.

At A draw the line AD, perpendicular to AB and equal to it (Prob. 7). Join DB. With B as centre, and radius BD, draw the arc DE till it meets AB produced in E. From E draw the line EF parallel to AC till it meets BC produced in F (Prob. 4). EBF is the triangle required.

As any rectilinear figure drawn on DB must equal the sum-areas of similar figures drawn on AB and AD, and as AD is equal to AB, the figure drawn on DB must be twice the area of the figure drawn on AB. The line EB is equal to the line DB, so the triangle EFB must be twice the area of the triangle ABC.

Note.—By this means any geometrical figure can be constructed of a given ratio to any similar geometrical figure, *e.g.* a quadrilateral, polygon, or circle could be constructed to

equal any number of times a given quadrilateral, polygon, or circle.

It has been shown (Prob. 116) that CD in Fig. 222 is the *mean proportional* between the lines AD and DB; by means of the right-angled triangle ABC we can extend the analogy.

The three triangles ABC, ADC, and CDB are similar; they are therefore equiangular. Consequently the line AB : AC : : AC : AD; therefore AC is the mean proportional between AB and AD.

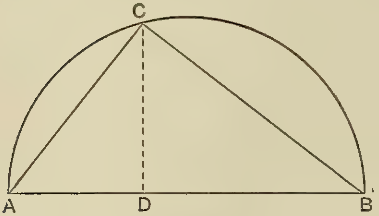


Fig. 222.

The line DB : BC : : BC : BA; therefore BC is the mean proportional between DB and BA.

The line DB : DC : : DC : DA; therefore DC is the mean proportional between DB and DA.

If three straight lines are proportionals, the rectangle contained by the extremes is equal to the square on the mean.

Problem 140.

To construct a square equal in area to a given rectangle ABDC.

Produce the line AB, and make BN equal to BD. Bisect AN in E (Prob. 1). With E as centre, and EA as radius, describe the semicircle AHN. Produce BD to H. BH is the mean proportional between AB and BD. On BH describe the required square BHKL (Prob. 26).

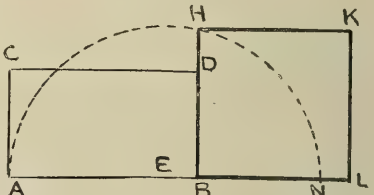


Fig. 223.

Problem 141.

On a given line AB, *to construct a rectangle equal in area to a given square* CDEF.

Find the third proportional to the lines AB and CD (Prob. 115), which will be the line AH. At A and B draw lines perpendicular to AB (Prob. 7), and make them equal to AH. Join KL. Then ABLK is the required rectangle.

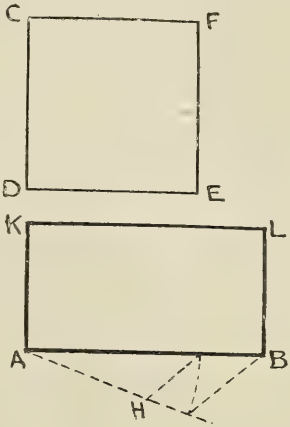


Fig. 224.

Problem 142.

To construct a parallelogram on a given line AB, *equal in area to a given parallelogram* CDEF.

Find the fourth proportional to the lines AB, DE, and DC (Prob. 113), which will be AH. Erect perpendiculars at A and B equal to AH (Prob. 7). Join KL, which will complete the rectangle ABLK required.

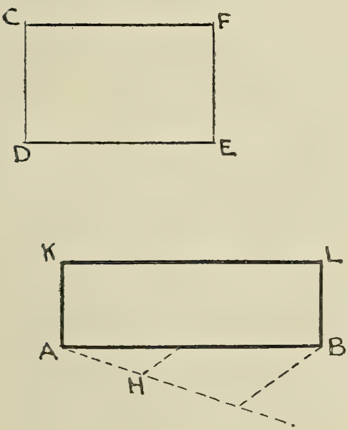


Fig. 225.

Problem 143.

To construct a square equal in area to a given triangle ABC.

First convert the given triangle into the rectangle ABED, by drawing the line CK perpendicular to AB (Prob. 5), and bisecting KC by the line DE parallel to AB (Prob. 1). Then

draw the lines AD and BE perpendicular to AB till they meet the line DE in D and E.

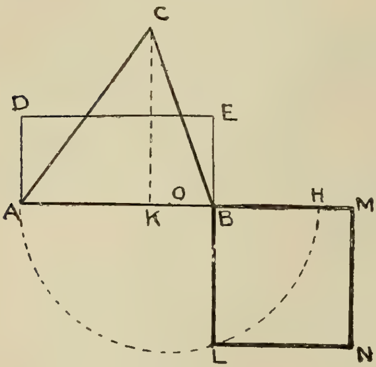


Fig. 226.

Produce the line AB, and make BH equal to BE. Bisect AH in O. With O as centre, and OA as radius, describe the semicircle ALH. Draw the line BL perpendicular to AB, which will be the mean proportional between AB and BE. On BL construct the square BLNM (Prob. 26), which will be the square required.

PROBLEM 144.

On a given line AB, *to draw a triangle equal in area to a given triangle* CDE.

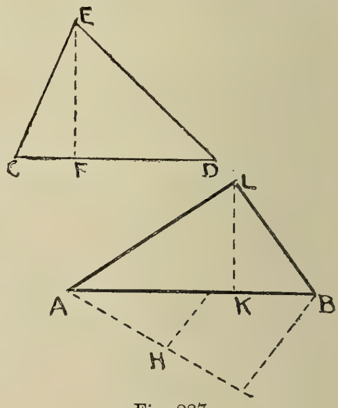


Fig. 227.

In the given triangle CDE draw the line EF perpendicular to CD (Prob. 5). Find the fourth proportional AH to the lines AB, CD, and EF (Prob. 113). At any point on the line AB, set up KL equal to AH, and perpendicular to AB. Join AL and LB. ALB is the triangle required.

PROBLEM 145.

To construct an equilateral triangle equal in area to a given triangle ABC.

On one side of the given triangle, CB for example, construct

an equilateral triangle CBD (Prob. 16). From the point A draw the line AE parallel to CB till it meets CD produced in E (Prob. 4). Bisect the line ED in H (Prob. 1). With H as centre, and HE as radius, draw the semicircle EKD. Find the mean proportional CK between the lines EC and CD (Prob. 116). On the line CK construct the required equilateral triangle.

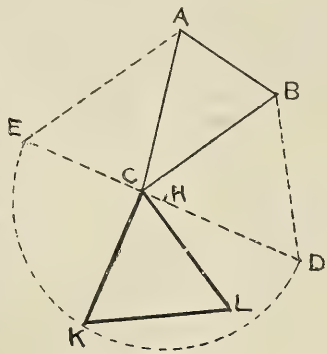


Fig. 228.

Problem 146.

Another method.

Let ABC be the given triangle. Convert it into an isosceles triangle (Prob. 124). Draw the line DE perpendicular to AB (Prob. 5). On the base AB construct the equilateral triangle ABH (Prob. 16). Bisect DE in G (Prob. 1). With G as centre, and GD as radius, draw the semicircle DFE. Find the mean proportional EF between the lines ED and EH (Prob. 116). Set off on ED, EK equal to EF, and draw the lines LK and KM parallel to AH and HB. LKM is the equilateral triangle required.

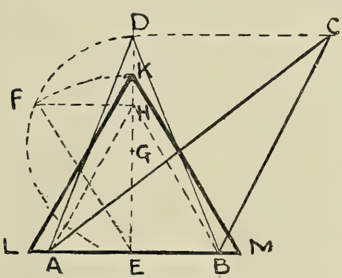


Fig. 229.

Problem 147.

To divide a given triangle ABC *into three equal divisions by lines parallel to one of its sides.*

Bisect one side, CB for example, in the point G (Prob. 1).

With G as centre, and GB as radius, draw the semicircle CFHB. Trisect the line CB in D and E (Prob. 10). Find the mean proportional CF between CB and CD (Prob. 116). Make CK equal to CF. Also find the mean proportional between CE and CB, viz. CH, and make CL equal to CH. Draw the lines KM and LN parallel to AB (Prob. 4), which will divide the given triangle ABC into three equal parts.

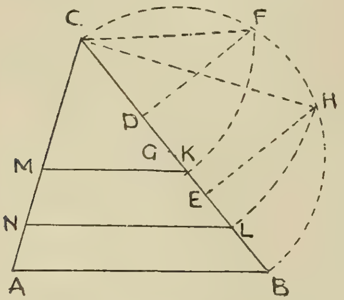


Fig. 230.

PROBLEM 148.

To construct a triangle one-fifth the area of a similar triangle ABC.

Produce the line AB to D, making DA equal to one-fifth of AB (Prob. 10). Bisect DB in F (Prob. 1), and with F as centre and FD as radius, draw the semicircle DEB. At A erect AE perpendicular to AB (Prob. 7). AE is the mean proportional between the lines AB and DA. Make HB equal to AE, and from H draw HK parallel to AC. HKB is the triangle required.

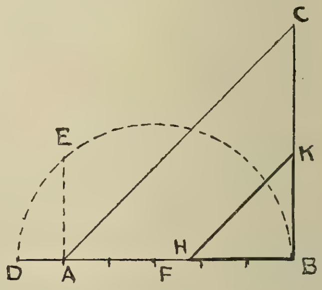


Fig. 231.

PROBLEM 149.

Construct a rectangle one-third the area of a similar rectangle ABC.

Produce the line AB to E, making AE equal to one-third

of AB (Prob. 10). Bisect EB in H (Prob. 1). With H as centre, and HE as radius, draw the semicircle EKB. From A draw the line AK perpendicular to AB (Prob. 7). AK is the

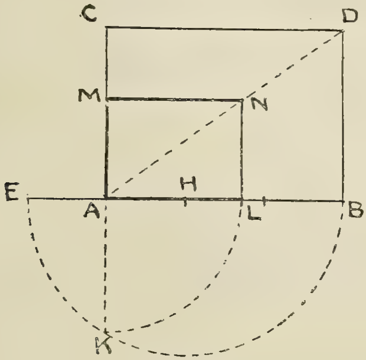


Fig. 232.

mean proportional between the lines EA and AB. Join AD. Make AL equal to AK. Draw LN perpendicular to AB till it meets AD in N. From N draw NM parallel to AL. ALNM is the rectangle required.

PROBLEM 150.

To construct a regular polygon three-fourths the area of a similar polygon, e.g. a hexagon.

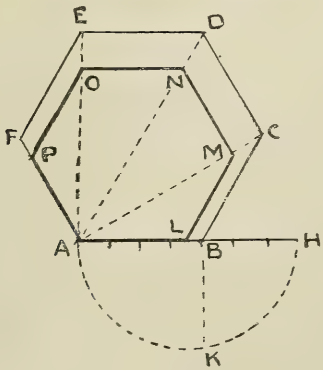


Fig. 233.

Let ABCDEF be the given hexagon. Produce AB to H,

making BH equal to three-fourths of AB (Prob. 10). Bisect AH (Prob. 1), and from its centre describe the semicircle AKH. Draw BK—the mean proportional between AB and BH—perpendicular to AB. Join the points E, D, and C with A. Set off AL equal to BK. Draw LM parallel to BC (Prob. 4), and MN, NO, and OP parallel to the lines CD, DE, and EF. Then ALMNOP is the hexagon required.

Problem 151.

To draw a circle three-fifths of the area of a given circle.

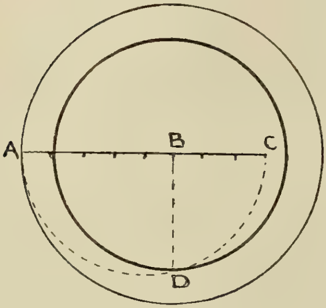


Fig. 234.

Let AB be the radius of the given circle. Produce AB to C, making BC three-fifths the length of AB (Prob. 10). Find the mean proportional BD between the lines AB and BC (Prob. 116). With B as centre, and BD as radius, draw the required circle.

Problem 152.

To divide a given circle into any number of equal parts, e.g. five.

Let AB be the radius of the given circle. Divide it into five equal parts (Prob. 10). Bisect AB (Prob. 1), and from its centre describe the semicircle ACB. Erect perpendiculars on each of the divisions of AB till they meet the semicircle in the points C, E, H, and K (Prob. 8). With A as centre, and the points C, E, H, and K as radii, draw the circles required.

AB : AC : : AC : AD (Fig. 222); therefore, AD being one-

fifth of AB, the circle through C is one-fifth of the given

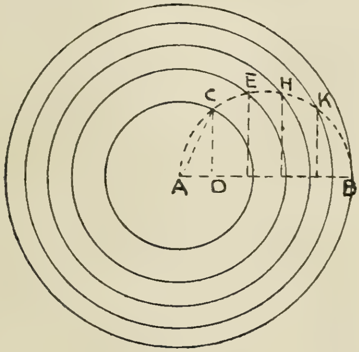


Fig. 235.

circle. Each of the other radii can be proved in the same way.

Problem 153.

To divide the area of a given circle into any number of equal parts which shall also have equal perimeters, e.g. five.

Draw the diameter AB, and divide it into five equal parts

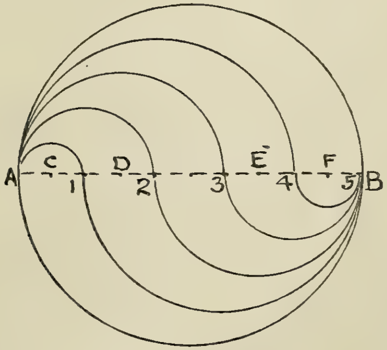


Fig. 236.

(Prob. 10). Bisect the 1st, 2nd, 4th, and 5th divisions in the points C, D, E, and F (Prob. 1). With C and F as centres, and

CA as radius, draw two semicircles. With 1 and 4 as centres, and 1A as radius, draw two semicircles. With D and E as centres, and DA as radius, draw two semicircles. With 2 and 3 as centres, and radius equal to 2A, draw two semicircles. The various semicircles will join each other in the points A, 1, 2, 3, 4, and 5, which will divide the area of the circle as required.

Problem 154.

To construct a triangle equal in area to a given circle ABC.

Draw any radius AD (Fig. 237), and make AE perpendicular to it (Prob. 7), and equal to the circumference of the circle in length (Prob. 192). Join DE. The triangle ADE will then be equal in area to the given circle ABC.

Note.—For practical purposes the circumference of a circle may be taken as equal to $3\frac{1}{7}$ times its diameter.

Fig. 237.

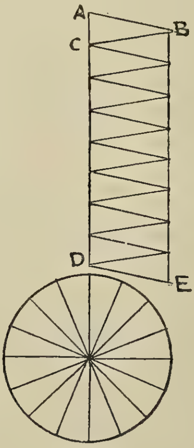


Fig. 238.

Problem 155.

To construct a parallelogram equal in area to a given circle.

Divide the given circle (Fig. 238) into any number of equal

triangles by drawing radii, say 16. Draw ABC to represent one of these triangles. Produce AC to D, and make AD eight times the length of AC. Draw BE parallel to AD and equal to it in length (Prob. 4). Join DE. Then ABDE will approximately equal the given circle in area.

Note.—If the longer sides of a rectangle were drawn equal to half the circumference, and the shorter sides equal to the radius, the rectangle would equal the circle in area. A circle can therefore be converted into a rectangle, square, rhomboid, or rhombus of the same area.

Table of Foreign Road Measures and their Equivalent in English Yards.

		English Yards.
Austria . . .	mile	8297
Bavaria . . .	„	8059
Belgium . . .	kilomètre	1094
Berne . . .	league	5770
China . . .	li	609
Denmark . . .	mile	8238
England . . .	„	1760
France . . .	kilomètre	1094
Germany . . .	mile	8101
Greece . . .	„	1640
Holland . . .	„	1094
India (Bengal) . .	coss	2000
Italy . . .	mile	2025
Netherlands . .	kilomètre	1094
Norway . . .	mile	12,182
Persia . . .	parasang	6076
Portugal . . .	mile	2250
Prussia . . .	„	8238
Russia . . .	verst	1167
Siam . . .	röeneng	4204
Spain . . .	mile	1522
Sweden . . .	„	11,690
Turkey . . .	berri	1827

CHAPTER X

PLAIN SCALES, COMPARATIVE SCALES, AND DIAGONAL SCALES

On a drawing representing a piece of machinery the scale is written thus: *Scale* $\frac{1}{4}$ *full size.* From this we know that every inch on the drawing represents 4 inches on the actual machine, so the relation between any part represented on the drawing and a corresponding part in the real object is as $1:4$ or $\frac{1}{4}$. This is called the *representative fraction.*

A drawing representing a building has drawn upon it a scale; *e.g.—Scale* $\frac{1}{4}$ *of an inch to a foot.* One-quarter of an inch is contained forty-eight times in 1 foot, so the R.F. is $\frac{1}{48}$.

On a large drawing showing a district the scale is written thus: R.F. $\frac{1}{1760}$. As there are 1760 yards to a mile, it is evident that every 3 feet on the drawing is equal to 1 mile on the land represented. This, of course, is a very large scale.

Our Ordnance Survey comprises one of 25 inches to a mile, which is used for small districts or estates; one of 6 inches to a mile, commonly used for maps of counties; and one of 1 inch = 1 mile.

The R.F. for the last would be $\frac{1}{63360}$.

mile.	yards.	feet.	inches.
1 =	1760 =	5280 =	63,360.

Problem 156.

To construct a scale 6 *inches long, showing inches and tenths of an inch.* Fig. 239.

Draw a line 6 inches long, and divide it into six equal parts,

each of which will represent 1 inch. At the end of the first inch mark the zero point, and from this point mark the inches to the right 1, 2, 3, 4, and 5. These are called primary divisions, and the amount by which they increase is called the value of the scale length.

The division left of the zero point has to be divided into ten equal parts. The best way to do this is to take a piece of paper and set off along its edge ten equal divisions of any convenient size (Fig. 239). Produce the perpendicular marking the division at the zero point, and arrange this piece of paper so as to fit in exactly between the end of the division and this perpendicular line. If we now draw lines parallel to the perpendicular at the zero point, they will divide the division into ten equal parts. These are called the secondary divisions, and they have the same zero point as the primary, and their numbers increase from this point by the value of their scale length.

This scale will measure inches to one decimal place. Supposing we wish to measure 3·6 inches, that is 3 primary and 6 secondary divisions. Place one point of the dividers on division 3 of the primary parts, and open them till the other point reaches the division 6 of the secondary divisions.

Problem 157.

To construct a scale of $\frac{1}{36}$, *or* 1 *inch to equal* 3 *feet.* Fig. 240.

Number of feet to be represented may be assumed at pleasure, say 18 feet.

$$36 : 18 :: 12 : x,$$

$$\text{whence } x = \frac{12 \times 18}{36} = 6 \text{ inches.}$$

Draw a line 6 inches long, and mark off each inch. Trisect each of these divisions by a piece of paper, as shown in Fig. 239. We now have the total length divided into eighteen equal parts.

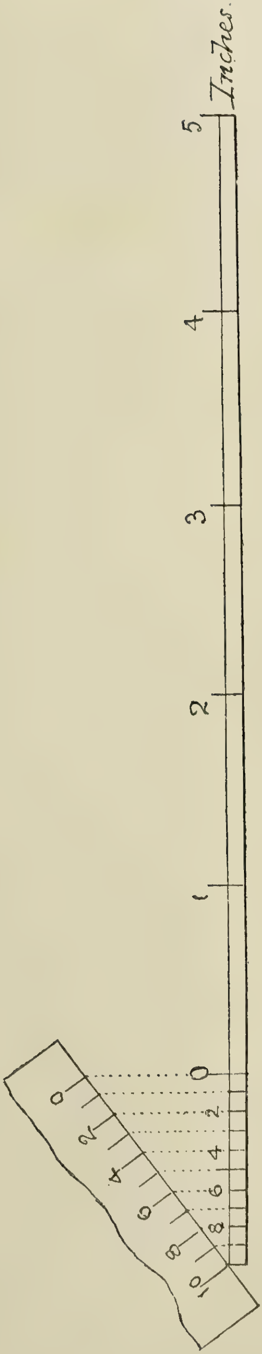


Fig. 239.

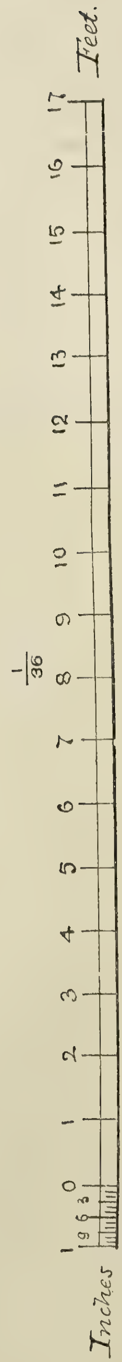


Fig. 240.

At the end of the first division mark the zero point, and from this point towards the right figure the primary divisions 1, 2, 3, etc.: these will represent feet. To the left of the zero point mark off twelve divisions: these will represent inches.

In this scale the numbers increase on each side of the zero point by unity.

This scale will measure feet and inches.

Problem 158.

To construct a scale with the R.F. $\frac{1}{288}$, *or* 1 *inch to equal* 8 *yards. To measure* 50 *yards.* Fig. 241.

$$288 : 50 :: 36 : x,$$

whence
$$x = \frac{36 \times 50}{288} = 6{\cdot}25 \text{ inches.}$$

Draw a line 6·25 inches in length, and divide it into five equal parts, *i.e.* 1·25 for each division. At the end of the first division mark the zero point. As each of the primary divisions is equal to 10 yards, we must figure them from the zero point to the right 10, 20, 30, and 40 yards.

The division to the left of the zero point we divide into ten equal divisions. Each of these secondary divisions represents 1 yard.

Problem 159.

To construct a scale, R.F. $\frac{1}{63360}$, *or* 1 *inch to a mile. To measure* 6 *miles.* Fig. 242.

If 1 mile = 1 inch, 6 miles = 6 inches.

Draw a line 6 inches long. Mark off each inch. At the end of the first division mark the zero point, and number the primary divisions to the right 1, 2, 3, 4, 5 miles. Divide the division to the left of the zero point into eight equal parts: these secondary divisions represent furlongs.

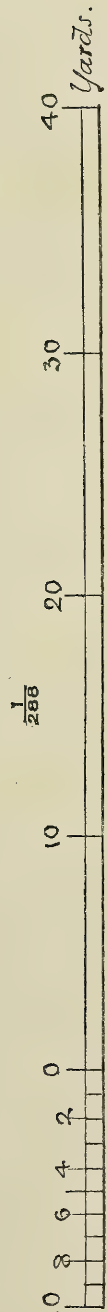


Fig. 241.

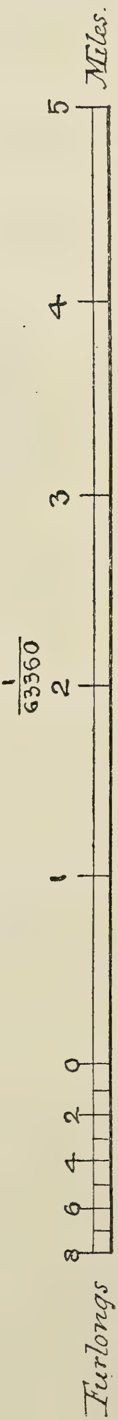


Fig. 242.

COMPARATIVE SCALES.

Problem 160.

On an old French map a scale of leagues is shown, as Fig. 243. Upon measuring this scale with an English scale, 30 leagues are found to coincide with 4 inches.

To construct a comparative scale of English miles. To measure 100 *miles.* Fig. 244.

A French league = 4262·84 English yards.

$$\underset{\text{leagues.}}{\overset{\text{French}}{30}} = \frac{4262{\cdot}84}{1760} \times \underset{\text{miles.}}{\overset{\text{English}}{30}}$$

$$\therefore \frac{4262{\cdot}84 \times 30}{1760} : 100 : : 4 : x,$$

whence $$x = \frac{4 \times 100 \times 1760}{4262{\cdot}84 \times 30} = 5{\cdot}5 \text{ inches nearly.}$$

Draw a line of this length, and place the zero point at the left-hand end and 100 at the other extremity. Divide this line into ten equal divisions: each of these primary divisions will represent 10 miles.

For the secondary divisions, set off one of the primary divisions to the left of the zero point, and divide it into ten equal divisions: each of these will represent 1 mile. The representative fraction of both the French and English scales will of course be the same.

Problem 161.

On a Russian map a scale of versts is shown, as Fig. 245, by measuring which by an English scale 120 versts = 4 inches.

To construct a comparative scale of English miles. Scale to measure 100 *miles.* Fig. 246.

A Russian verst = 1167 English yards.

Fig. 243.

Fig. 244.

Fig. 245.

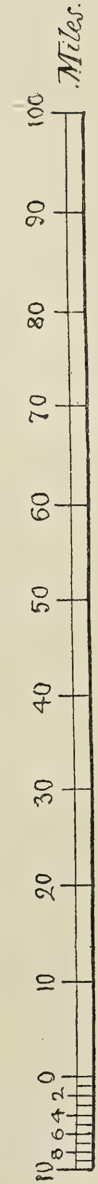


Fig. 246.

$$\overset{\text{versts.}}{120} = \frac{\overset{\text{yards.}}{1167} \times \overset{\text{miles.}}{120}}{1760}$$

$$\therefore \frac{1167 \times 120}{1760} : 100 :: 4 : x,$$

whence $$x = \frac{4 \times 100 \times 1760}{1167 \times 120} = 5 \text{ inches nearly.}$$

Draw a line of this length, and divide it into ten equal divisions: each of these primary divisions will represent 10 miles. Place the zero point at the left-hand end of the line, and figure the divisions towards the right 10, 20, 30, etc. Set off one of the primary divisions to the left of the zero point, and divide it into ten equal divisions: each of these will represent 1 mile.

DIAGONAL SCALES.

In the preceding scales we have only primary and secondary divisions, and if we wish to measure a fractional proportion of a secondary division, we cannot do it with any accuracy; but by means of a *diagonal scale* we are enabled to measure hundredths of primary divisions, as will be seen from the following scale.

Problem 162.

To construct a diagonal scale 6 *inches long, to measure inches, tenths of inches, and hundredths of inches.* Fig. 247.

Draw a rectangle ABCD 6 inches long and about $1\frac{1}{4}$ inches wide, and divide it into six equal parts. At the end of the first division from A fix the zero point, and to the right of this figure each division 1, 2, 3, 4, and 5. Divide AC into ten equal parts, and figure them from A towards C, then draw lines parallel to AB from one end of the scale to the other. Divide A0 into ten equal divisions, and figure them from 0 towards A. Join 9 to C, and from each of the other divisions between A

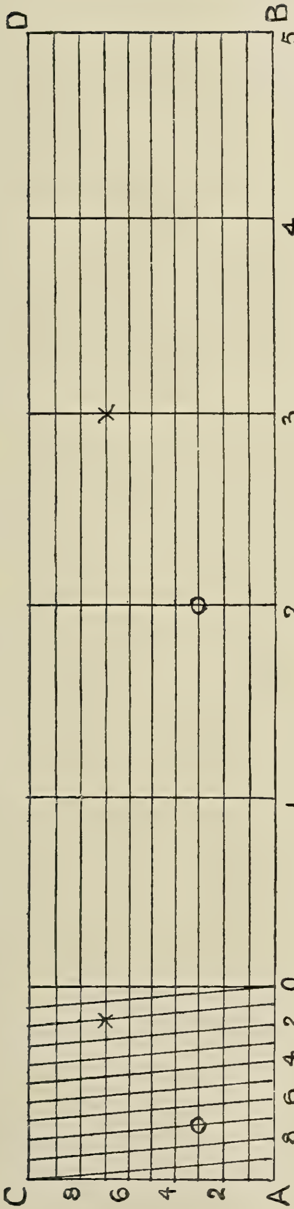


Fig. 247.

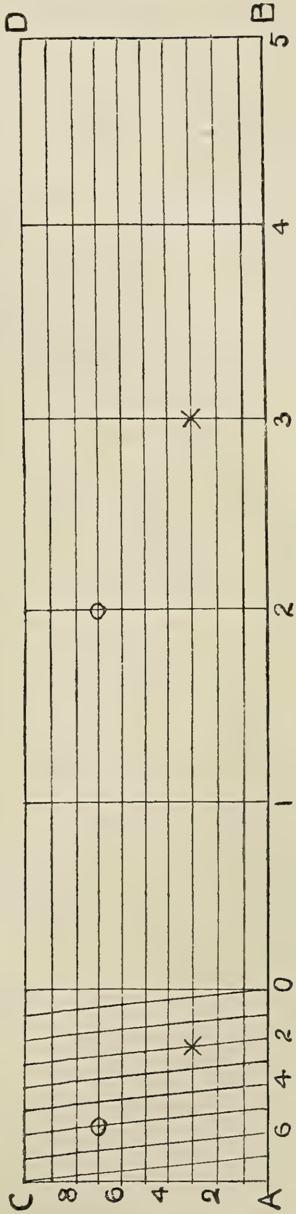


Fig. 248.

and 0 draw lines parallel to 9C. *Note.*—The divisions between A and C are called diagonal divisions.

To take off from this scale a measurement equal to 2·73 inches, we place one point of the dividers on the primary division figured 2, and the other on the secondary division figured 7, but both points must be on the line that is figured 3 on AC. The points are marked by a small circle on the scale.

To take off 3·17 inches, place one point of the dividers on the primary division 3, and the other on the intersection between the secondary division 1 and the line 7 on AC. These points are shown by two crosses on the scale.

Problem 163.

To construct a diagonal scale showing miles, furlongs, and chains, to show 6 *miles,* R.F. $= \frac{1}{63360}$. Fig. 248.

1 mile = 8 furlongs.
1 furlong = 10 chains.
The length of scale $= \frac{1}{63360}$ of 6 miles = 6 inches.

In this scale there will be six primary, eight secondary, and ten diagonal divisions.

Construct a rectangle ABCD 6 inches long and about $1\frac{1}{4}$ inches wide, and divide it into six primary divisions. Place the zero point 0 at the end of the first division from A, and divide A0 into eight secondary divisions, figured from 0 towards A. Divide AC into ten equal divisions, and figure them from A to C. Join the secondary division figured 7 to the point C, and from each of the other secondary divisions draw lines parallel to 7C, thus completing the scale.

To take off from this scale 2 *miles,* 5 *furlongs, and* 7 *chains,* take one point on the primary division 2, and the other where the line from the secondary division 5 intersects the diagonal division 7. These two points are marked by small circles on the scale.

To take off 3 *miles, 2 furlongs, and 3 chains,* one point will be on the primary division 3, and the other where the secondary division 2 intersects the diagonal division 3. These points are marked by a small cross on the scale.

Problem 164.

The diagonal scale generally found in instrument-boxes is shown in Fig. 249.

It consists of two diagonal scales. In one, the distance between the primary divisions is half an inch, and in the other a quarter of an inch.

There is a small margin on each side of the scale for figures: on one side the half inches are figured, and the quarter inches on the other.

One primary division at each end is divided into ten secondary divisions, and there are ten diagonal divisions drawn from one end of the scale to the other.

The primary divisions being taken for units, to set off the numbers 5·36 by the diagonal scale. This measurement is shown by two crosses on the scale.

If we reckon the primary divisions to stand for tens, the dimension would have one place of decimals, *e.g.* take off 36·4 from the diagonal scale. These points are shown on the scale by two small circles.

The primary divisions being hundreds, to take off 227. This dimension is shown on the scale by two small squares.

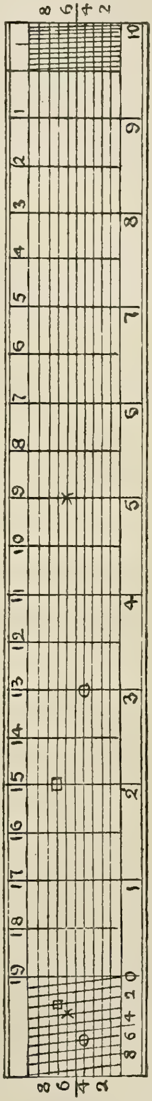


Fig. 249.

General Rule.

To take off any number to three places of figures from a diagonal scale.

On the parallel indicated by the third figure, measure from the diagonal indicated by the second figure to the vertical line indicated by the first.

PROPORTIONAL SCALES.

These are used for enlarging or reducing a drawing to a given proportion: three varieties are here illustrated.

The simplest form is that shown in Fig. 250. Suppose we wish to *enlarge* a drawing in the proportion of 3 : 1.

Draw the line AB of convenient size, to suit the measure-

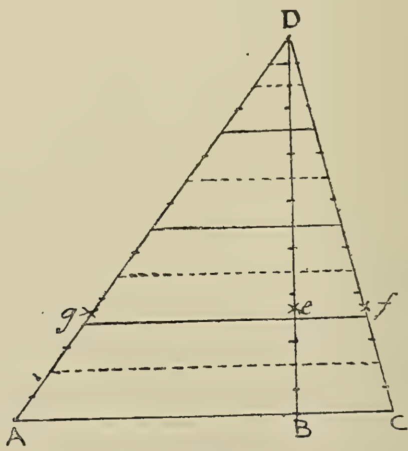


Fig. 250.

ments on the drawing, and produce it to C; make BC one-third of AB. On AB erect the perpendicular BD any length, and join AD and DC. Divide BD into any number of equal parts, and draw lines parallel to AB. These lines are simply a guide to enable the measurements to be made parallel to the base—*e.g.* on placing a measurement from the original drawing on

the scale, we find it occupies the position of *ef*; the distance between *e* and *g* will then give the length of the measurement to the enlarged scale, *i.e.* in the proportion of 3 : 1. We proceed in the same manner with every measurement we wish to enlarge.

Should we wish to *reduce* a drawing in the same proportion, viz. 1 : 3, the original measurements would be placed on the left-hand side of the scale, and the required proportion taken from the right-hand side.

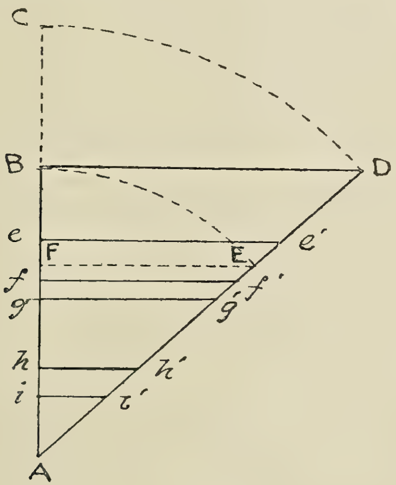


Fig. 251.

In Fig. 251 we have a series of measurements—A*i*, A*h*, A*g*, etc.—which we wish to *enlarge*, say in the proportion of 3 : 2.

Draw the line AB any convenient length to suit size of drawing. From B draw BD perpendicular to AB. Produce AB to C, and make BC equal to half of AB. With centre A, and radius AC, draw an arc till it meets BD in D; and join DA. From each of the points, *i*, *h*, *g*, *f*, etc., draw lines parallel to BD. The distances A*i′*, A*h′*, A*g′*, etc., will then give the original measurements to the enlarged scale of 3 : 2.

To *reduce* the original drawing in the same proportion, *i.e.* 2 : 3. With A as centre, and radius AB, draw the arc BE. From E draw the line EF parallel to BD. AF will then repre-

sent AE reduced in the proportion of 2 : 3, and so on with any other measurement that we may require.

Fig. 252 is the problem generally used for dividing a line into

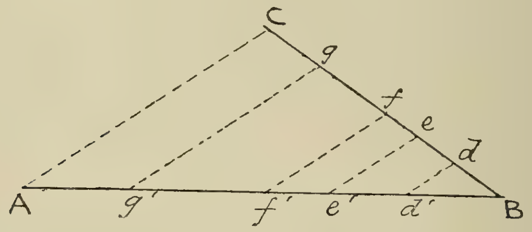


Fig. 252.

a given number of equal parts, but it can also be used for enlarging or reducing a drawing to a given scale.

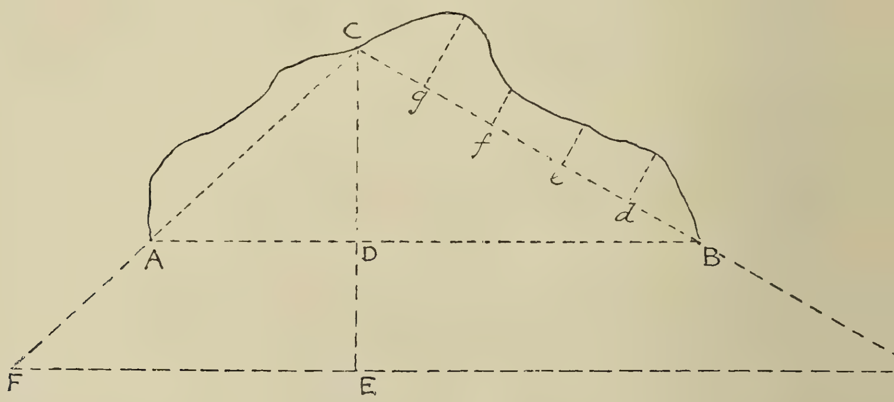


Fig. 253.

Let Fig. 253 represent a portion of the plan of a field which we wish to draw to an enlarged scale, say in the proportion of 5 : 3.

First enlarge the triangle ABC. This can be done by drawing the line CE at right angles to AB, making DE equal to two-thirds of CD, and drawing FH through E parallel to AB till it meets CA and CB produced in F and H. Then CFH will be

the enlarged triangle, bearing the same proportion to ABC as 5 : 3.

Divide the line CB into any number of equal parts in the points *g*, *f*, *e*, and *d*, and draw lines at these points perpendicular to CB.

Draw the line AB (Fig. 252) any convenient length, and from B draw BC at any angle. Make BC three-fifths the length of AB, and join CA. From B set off B*d* equal in length to the perpendicular at *d* (Fig. 253). Make *de* equal to the length of the perpendicular at *e*, and so on with all the other points. From each of these points draw lines parallel to AC. The divisions on AB will then represent the corresponding divisions on CB to the enlarged scale, *i.e.* B*d′* bears the proportion to B*d* as 5 : 3, and so on with all the other points.

Divide the corresponding side of the enlarged triangle CH (Fig. 253) into the same number of equal parts, and erect perpendiculars at these points. Make these perpendiculars equal in height to the corresponding divisions on AB (Fig. 252). Join the tops of the perpendiculars by an irregular curve, which will represent the curve on CB enlarged in the proportion of 5 : 3. Proceed in the same manner with the side AC.

Note.—The above problems could, of course, be used for any other proportions than those chosen.

A very interesting proportional scale could be constructed from the principle of Euclid, 1st Book, Prob. 47.

Let ABC (Fig. 254) be a right-angled triangle. Construct a proportional scale equal to the three sides of this triangle. The point F may be in any convenient position above the line DE.

Let *ab* represent a measurement from a given figure, then *ca* and *bd* will represent corresponding measurements of similar figures, the combined area of which would be equal to the area of the given figure. Also *ca* would represent the measurement of a figure, the area of which would be the difference between the areas of two similar figures, the corresponding measurements of which are represented by the lines *ab* and *bd*. In the same

way *bd* would represent a figure whose area is equal to the

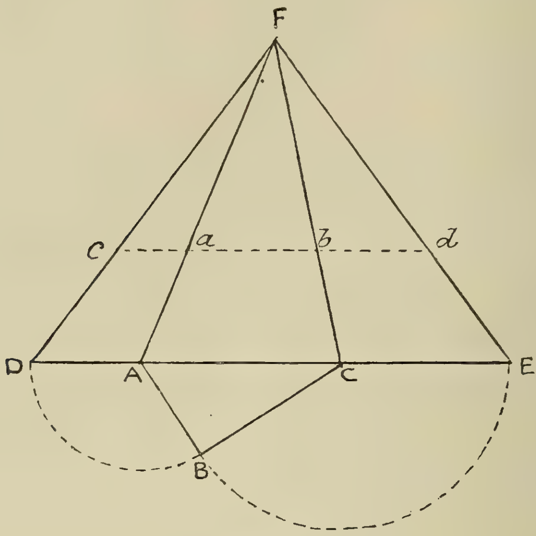


Fig. 254.

difference of the similar figures represented by the corresponding lines *ab* and *ca*.

Note.—These similar figures may be of any conceivable shape.

CHAPTER XI

INSTRUMENTS FOR MEASURING ANGLES, ETC.

A *protractor* is an instrument used for measuring or setting off angles; it may be either semicircular or rectangular in shape, as shown in Fig. 255. The point C marks the centre from which

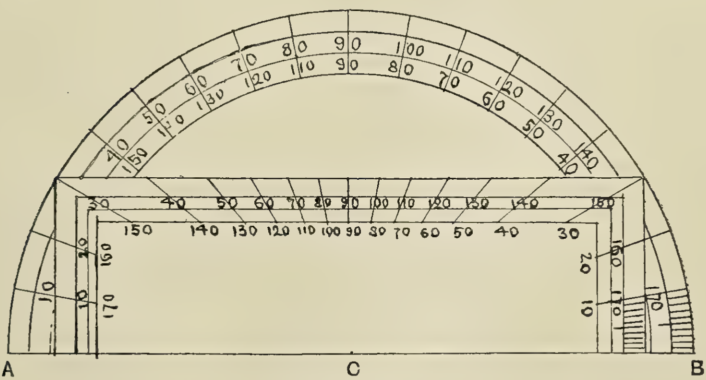


Fig. 255.

the radiating lines are drawn, and corresponds with the centre of the circle.

The degrees are numbered in primary divisions, equal to ten degrees each, on the outside line from A; and on the inside line from B. In the actual instrument each of these primary divisions is subdivided into ten secondary divisions, each of which represents one degree. Only one of these is divided in the figure.

Problem 165.

To construct a scale of chords.

A *scale of chords* is constructed in the following manner (Fig. 256). Draw the lines AC and CD perpendicular to each other (Prob. 5). With C as centre, draw any quadrant AED, and divide the arc into degrees (only the primary divisions are shown in the figure). Join AD. With A as centre, and each

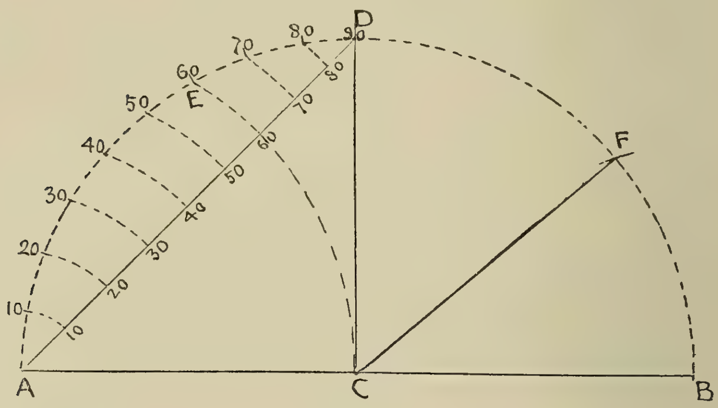


Fig. 256.

of the primary divisions as radii, draw arcs cutting the chord AD, which will form the scale of chords.

To use this scale in setting off an angle—for example, to draw a line that will make an angle of 40° with line CB (Fig. 256).

With C as centre, and radius equal to A60 on the scale of chords, draw an arc BFD. With a pair of dividers, take the distance A40 from the scale, and set it off on the arc BF from B. Join FC. Then FCB will be the angle of 40° required.

Note.—60° is always equal to the radius of the quadrant from which the scale of chords is constructed.

PROBLEM 166.

To construct any angle without a protractor.

Draw CD perpendicular to AB (Prob. 5). With C as centre, and CA as radius, draw the semicircle ADB. Trisect the angle DCB in E and F (Prob 14). Trisect the angle ECB in H and K (Prob. 15). Bisect FK in L (Prob. 2). Then $DE = 30°$,

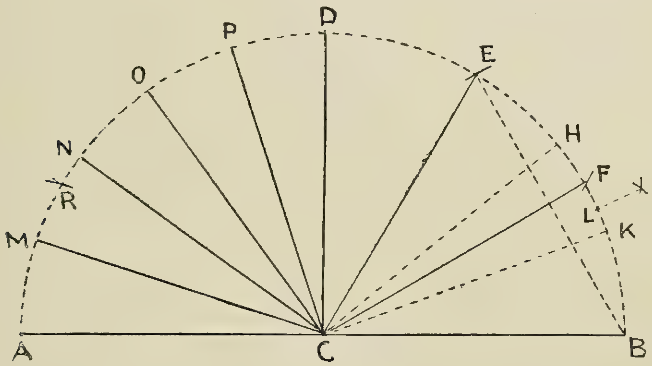


Fig. 257.

$EH = 20°$, $HF = 10°$, and $FL = 5°$. Therefore between D and B we can construct any angle that is a multiple of 5°.

Divide the angle ACD into five equal parts by the lines M, N, O, and P (Prob. 121). From A set off AR equal to DE. As $AN = 36°$ and $AR = 30°$, $AN - AR = 6°$, $MR = 12°$, $AM = 18°$, and $RO = 24°$. Therefore between A and D we can construct any angle that is a multiple of 6°.

If we subtract the multiples of 5° from those of 6° we can obtain any desired angle, *e.g.*—

$$6 - 5 = 1°$$
$$12 - 10 = 2°$$
$$18 - 15 = 3°$$
$$24 - 20 = 4°$$
$$30 - 25 = 5°$$

etc. etc. etc.

All the regular polygons, with the exception of two—the heptagon and undecagon—can be constructed with angles that are multiples of 5° or 6°—

If the polygon is to be inscribed in a circle, the angle would be set off at the centre of the circle; but if one side of the polygon is given, the angle would be set off externally, as shown in Fig. 94, Chap. III.

The exterior angle of a Pentagon	is 72°	a multiple of 6°.
,, ,, Hexagon	,, 60°	,, 6°.
,, ,, an Octagon	,, 45°	,, 5°.
,, ,, a Nonagon	,, 40°	,, 5°.
,, ,, Decagon	,, 36°	,, 6°.
,, ,, Duodecagon	,, 30°	,, 5° and 6°.

THE SECTOR.

The *Sector* is an instrument of great utility for facilitating the work of Practical Geometry. It consists of graduations on the two radii of a foot-rule, and it is used by measuring the arc between the graduations. Hence its name. The legs can be opened to contain any angle up to a straight line.

In the illustration (Fig. 258) only the lines mostly used in Practical Geometry are shown: viz. *line of lines,* marked L on each leg; a pair of *lines of chords,* marked C; and a *line of polygons,* marked POL, on the inner side of each leg.

The *sectoral lines* proceed in pairs from the centre of the hinge along each leg, and although the scales consist of two or three lines each, all measurements must be made on the inner lines of each scale, *i.e.* the lines that radiate from the centre.

When the measurement is confined to a line on one leg of the sector, it is called a *lateral distance;* but when it is taken from a point on a line on one leg to a similar point on a corresponding line on the opposite leg, it is called a *transverse distance.*

Simple proportion.—Let AB and AC (Fig. 259) represent a pair of sectoral lines, and BC and DE two transverse measurements

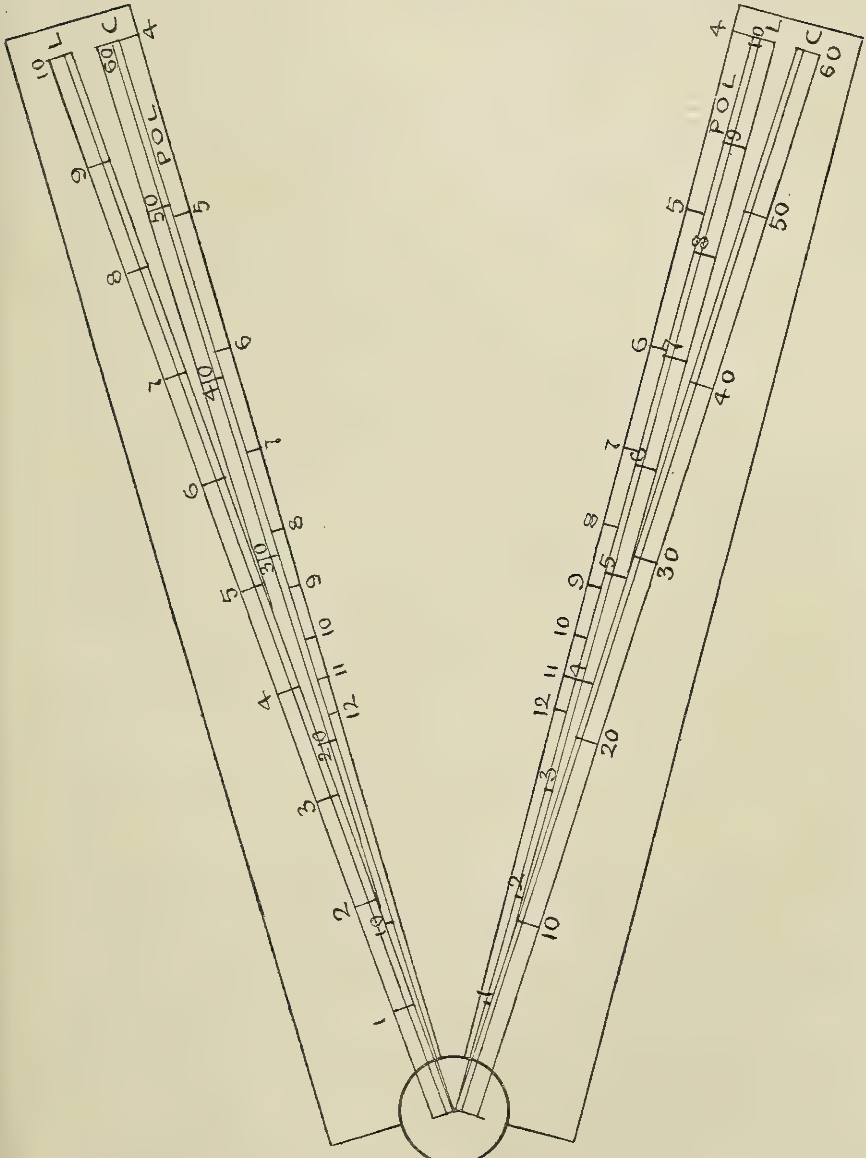


Fig. 258.

taken between this pair of lines; then AB is equal to AC, and AD to AE, so that AB : AC : : AD : AE, and the lines AB : BC : : AD : DE.

THE LINE OF LINES.

The primary divisions only are shown in the illustration; in the real instrument, each of these is subdivided into ten secondary divisions.

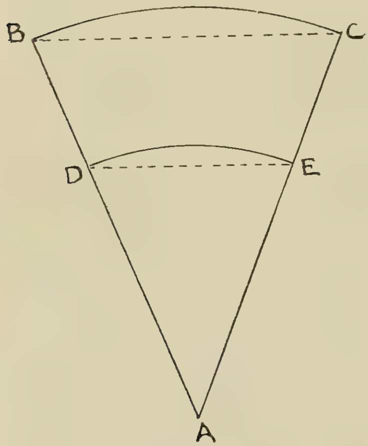


Fig. 259.

To find the fourth proportional to three given lines.—From the centre measure along one leg a lateral distance equal to the first term; then open the sector till the transverse distance between this point and a corresponding point on the other leg is equal to the second term; then measure from the centre along one leg a lateral distance equal to the third term; the transverse distance from this point to a corresponding point on the opposite leg will then give the fourth term.

Example.—To find the fourth proportional to the numbers 3, 4, and 9. From the division marked 3, which is the first term, open the sector till the distance between this point and the corresponding division on the other leg is equal to 4 divisions: this will be the second term. Then 9 being the third term, the transverse distance between the corresponding divisions at this point will give the fourth term, viz. 12.

To find the third proportional to two given lines or numbers.—Make a third term equal to the second, then the fourth term will give the required result.

To bisect a given line.—Open the sector till the transverse

distance between the end divisions, 10 and 10, is equal to the given line; then the transverse distance between 5 and 5 will bisect the given line.

To divide a given line AB *into any number of equal parts*—for example, eight (Fig. 260).

When the number of parts are a power of 2, the division is best performed by successive bisections. Thus, make AB

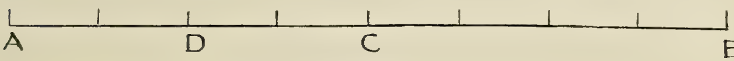


Fig. 260.

a transverse distance between 10 and 10, then the distance between 5 and 5 will give AC = half AB. Then make the transverse distance between 10 and 10 = AC, the distance between 5 and 5 will then give AD = one quarter of AB. By repeating the operation each quarter will be bisected, and the given line divided into eight equal parts as required.

When the number of divisions are unequal,—for example, seven (Fig. 261),—make the transverse distance between 7 and 7

Fig. 261.

equal to the given line AB; then take the distance between 6 and 6, which will give AC. The distance CB will then divide the line into seven equal parts.

How to use the sector as a scale.

Example.—*A scale of* 1 *inch equal* 5 *chains.*

Take one inch on the dividers, and open the sector till this forms a transverse distance between 5 and 5 on each *line of lines;* then the corresponding distances between the other divisions and subdivisions will represent the number of chains and links

indicated by these divisions: for instance, the distance between 4 and 4 represents 4 chains, 6·5 = 6 chains 50 links, 3·7 = 3 chains 70 links, etc.

Note.—1 chain is equal to 100 links.

To construct a scale of feet and inches, in which $3\frac{1}{2}$ *inches shall equal* 20 *inches.*

Make the transverse distance between 10 and 10 equal to $3\frac{1}{2}$ inches; then the distance between 6 and 6 will equal 12 inches. Make AB (Fig. 262) equal to this length. Bisect this

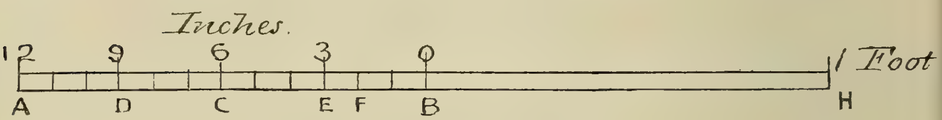


Fig. 262.

distance in C, as described for Fig. 260; then bisect AC and CB in D and E in the same manner. Take the transverse distance between 5 and 5, which will give AF; EF will then trisect each of the four divisions already obtained. AB will then be divided into twelve divisions, which will represent inches. Produce the line AB to H, and make BH equal to AB. BH will then represent one foot.

How the sector may be used for enlarging or reducing a drawing.

Example.—*Let* ABC (Fig. 263) *represent three points in a drawing, let it be required to reduce this in the proportion of* 4 *to* 7.

Make the transverse distance between 7 and 7 equal to AB; then take the distance between 4 and 4, and make DE (Fig. 264) equal to this length. Also make the distance between 7 and 7 equal to AC; then take the distance between 4 and 4, and from D as centre, with this distance as radius, describe an arc. In

the same manner make the distance between 7 and 7 equal to BC; then with a radius equal to 4, 4, describe another arc from

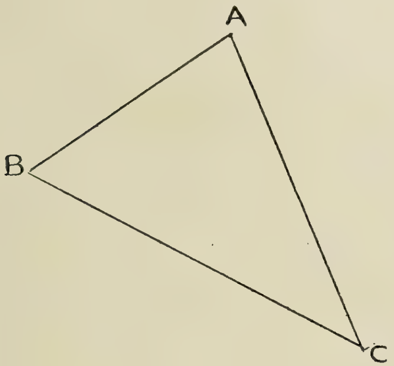


Fig. 263.

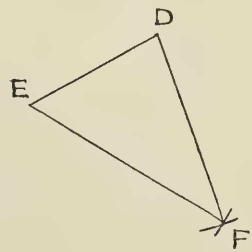


Fig. 264.

E, cutting the other arc in F. Join EF and DF. Then DEF will be a reduced copy of ABC, in the proportion of 4 : 7 as required.

To enlarge a drawing in the proportion of 7 *to* 4.

In this instance the sector would be opened so that the transverse distance between 4 and 4 should represent the original measurements, while those required for the copy would be taken between 7 and 7.

The Line of Chords.

In the scale of chords already described (Prob. 165) we are limited to one radius in setting off angles—viz. a radius equal to the 60° marked on the scale; in the double *line of chords* on the sector there is no such limitation—we can set off any radius equal to the transverse distance between the two points 60 and 60, from their nearest approach to each other up to the fullest extent the opening of the sector will admit of.

To construct an angle of 50°.

Open the sector at any convenient distance. Take the transverse distance between the points 60 and 60, and construct an arc with this radius. Let AB (Fig. 265) represent this radius. Now take the transverse distance between 50 and 50, and set it off from B on the arc, which will give the point C. Join AB and AC. Then BAC will be 50° as required.

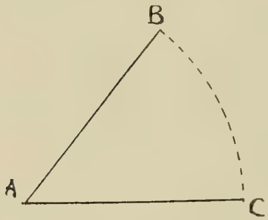


Fig. 265.

A greater angle than 60° cannot be taken from the sector with one measurement; if the angle to be measured is more than 60°, additional measurements must be taken.

On an arc $1\frac{1}{2}$ *inches in radius, to construct an angle of* 125°.

Make the transverse distance between the points 60 and 60 $1\frac{1}{2}$ inches. Let AB (Fig. 266) represent this distance. Describe an arc with AB as radius. Take the distance between the

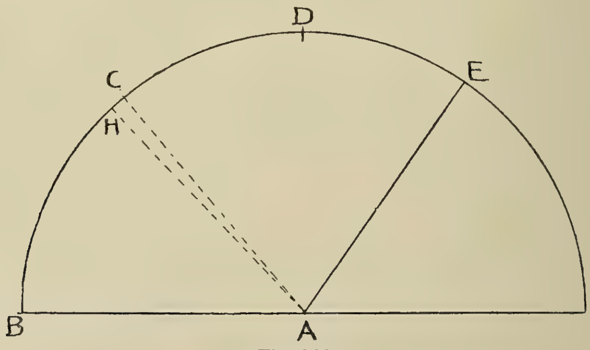


Fig. 266.

points 50 and 50 from the sector, and set it off on the arc from B to C. Also take the distance from 40 to 40, and set it off from C to D. Then take the distance between 35 and 35, and set it off from D to E. Join EA. Then the angle BAE will be 125°. $50° + 40° + 35° = 125°$.

Construct an angle of $3°$ on the same arc.

With the sector open at the same angle as before, take the transverse distance between the points 47 and 47, and set it off on the arc from B to H. Join HA and CA. Then HAC will be $3°$ as required. $50° - 47° = 3°$.

The Line of Polygons.

This pair of lines is used for dividing a circle into any number of equal parts between four and twelve, by joining which the regular polygons are formed. The transverse distance between the points 6 and 6 is always used for the radius of the circle to be divided; because the radius of a circle containing a six-sided figure, *i.e.* a hexagon, is always equal to one side of the figure.

Open the sector till the transverse distance between 6 and 6 is equal to the radius of the circle; then the distance between the points 4 and 4 will divide the circle into four equal parts, the distance between 5 and 5 into five equal parts, and so on up to twelve.

If it be required to construct a polygon on a given straight line, open the sector till the transverse distance between the numbers answering to the number of sides of the required polygon shall equal the extent of the given line, then the distance between the points 6 and 6 will give the radius of the circle to be divided by the given line into the required number of equal parts.

Example.—*On a given line* 1 *inch in length, to construct a heptagon.*

Open the sector till the transverse distance between the points 7 and 7 shall equal 1 inch; the distance between the points 6 and 6 will then give the radius of a circle, to which the given line will form seven equal chords.

CHAPTER XII

PLANS OF SURVEYS AND MEASUREMENT OF AREAS

PLANS of land are drawn from distances measured in two directions: viz. *direct distances,* or distances measured along the line AB (Fig. 267); and *lateral distances, i.e.* distances measured

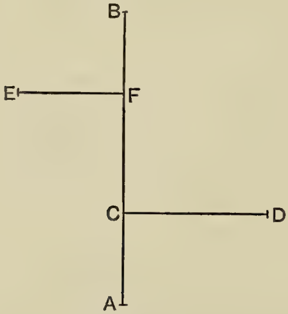


Fig. 267.

Fig. 268.

on lines perpendicular to AB, as CD to the right, and EF to the left of AB. These distances are recorded in a field-book, from which the plan is plotted to scale on a drawing. Let us suppose A to be the starting or zero point; from A to C = 8 yards, from A to F = 18 yards, and from A to B = 25 yards. These dimensions would be placed in a column in their consecutive order (Fig. 268). Then let CD represent 12 yards (this would be entered to the right-hand side of 8, because 8 represents the position of C), and let EF = 9 yards (this would be entered to the left of 18, the figure corresponding to F). From these figures we could draw a plan of the piece of land measured.

Draw the plan of a field from the following dimensions:—

$\frac{1}{8}$ of an inch = 10 yards, R.F. $\frac{1}{2880}$.

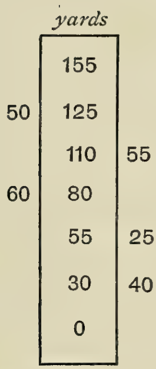


Fig. 269.

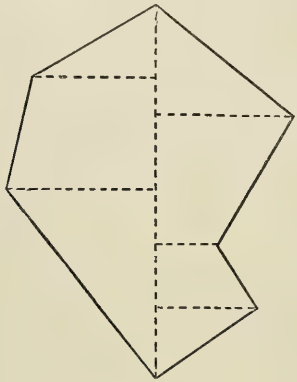


Fig. 270.

This plan is plotted to a scale of $\frac{1}{8}$ of an inch to 10 yards in Fig. 270.

Suppose we wish to make the plan in its greatest dimensions 6 inches long, we must first construct a scale of yards. Let the scale length = 100 yards.

$$\text{Then } \overset{\text{yds.}}{155} : \overset{\text{ins.}}{6} :: \overset{\text{yds.}}{100} : x,$$

$$\text{whence } x = \frac{100 \times 6}{155} = \overset{\text{inches.}}{3{\cdot}9} \text{ nearly.}$$

Draw a line this length, and divide it into ten equal parts; each of these primary parts will represent 10 yards. Divide the left-hand division into ten equal parts to represent yards, as shown in Fig. 241. The R.F. of this scale is $\frac{1}{930}$.

Problem 167.

To find the area of a piece of land.

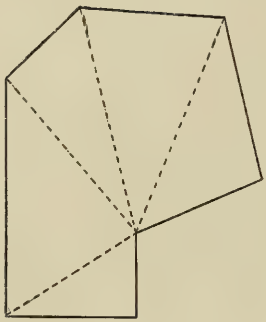


Fig. 271.

The field we have just plotted represents an irregular polygon, which could be transformed into an equivalent triangle, as Prob. 129; but the more practical method would be to divide it into triangles, as shown by dotted lines in Fig. 271.

The sum of the triangles would give the area of the piece of land.

To find the area of a triangle, multiply its base by half its altitude.

CHAPTER XIII

SIMILAR FIGURES

Similar figures have their angles equal and their corresponding sides proportional.

All regular figures—such as *equilateral triangles, squares,* and *regular polygons*—are similar. Other quadrilateral figures—triangles and irregular polygons—can be constructed similar to given ones by making their angles equal.

Problem 168.

To construct within a given triangle ABC, *and equidistant from the sides of it, a similar triangle, the base of which is equal to the given line* D. Fig. 272.

Bisect the angles BAC and ACB by lines meeting at the centre E (Prob. 13). Join EB. On the line AB set off AF equal to the given line D. From F draw a line parallel to AE till it cuts EB at G (Prob. 4). From G draw a line parallel to BC till it cuts EC at H. From H draw a line parallel to AC till it cuts EA at K. Join KG. KGH will be the similar triangle required.

Problem 169.

To construct about a given triangle ABC, *and equidistant from its sides, a similar triangle, the base of which is equal to a given line* L. Fig. 272.

Set off on the base AB produced, AN equal to the given line

L. From N draw a line parallel to EA (Prob. 4) till it meets EB produced at O. From O draw a line parallel to AB till it

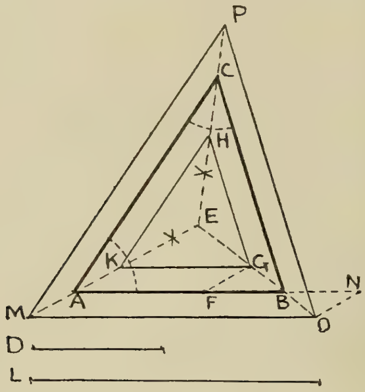


Fig. 272.

meets EA produced at M. From M draw a line parallel to AC till it meets EC produced at P. Join PO. Then MOP will be the similar triangle required.

PROBLEM 170.

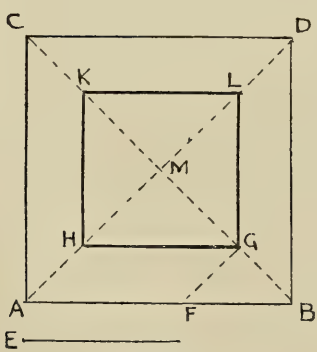


Fig. 273.

To construct within a given square ABCD, *and equidistant from it, a square, one side of which is equal to the given line* E.

Draw the diagonals AD and CB. From A on AB set off AF equal to the given line E. From F, parallel to AD, draw a line till it meets CB at G (Prob. 4). With M as centre, and radius MG, set off the points H, K, and L, and join GH, HK, KL, and LG. Then HGLK will be the square required.

Problem 171.

To construct a triangle similar to a given triangle CDE, *and having its perimeter equal to a given straight line* AB.

On the given line AB construct a triangle ABF similar to the given triangle CDE, by making the angles at A and B equal at the angles at C and D respectively. Bisect the angles at A and B by lines meeting at G. From G draw a line parallel to FB till it meets AB at L (Prob. 4); and from G draw another line parallel to AF till it meets AB at H. Then HLG will be the similar triangle required.

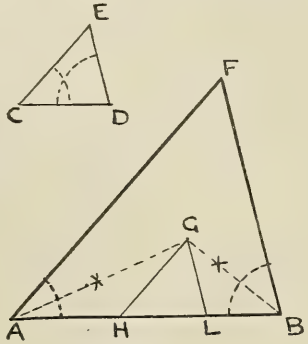


Fig. 274.

See also Figs. 216, 217, 221, 231, 232, and 233, as well as the problems in Chapter XIV.—"*Principles of Similitude.*"

CHAPTER XIV

PRINCIPLES OF SIMILITUDE

Problem 172.

Draw a rectangle ABCD, and join each angle to any point E. Bisect EB in G, EA in F, ED in K, and EC in H (Prob. 1). Join FG, GK, KH, and HF, then FGHK will be a rectangle with sides one-half the length of the rectangle ABCD. If we draw the diagonals BC and GH, we shall find they are

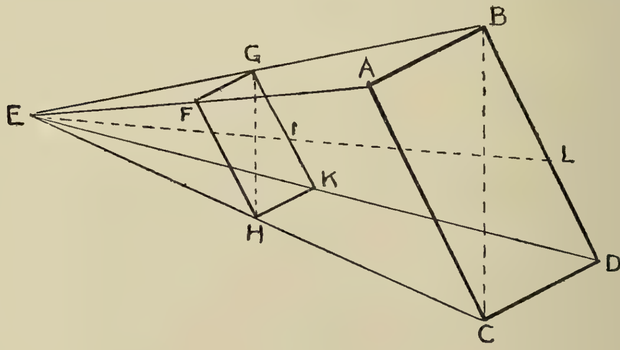


Fig. 275.

parallel to each other, and that BC : GH as 2 : 1. If we take any point L in BD and draw it to point E, it will intersect GK in M, and will divide GK in the same proportion as BD is divided.

On the principle of this problem, we can draw a figure similar to a given figure, and having any proportion desired, *e.g.* —

If we wish to draw a rectangle having sides equal to one-

third of a given rectangle, we should trisect the lines drawn from the angles of the given rectangle to E.

The point E is called the *centre of similitude* of the two figures ABCD and FGHK, which in this instance are said to be in *direct similitude.* The following problem shows the principle of *inverse similitude.*

Problem 173.

To draw a trapezium similar to a given trapezium ABCD, *with sides two-thirds the length of the given trapezium, in* INVERSE SIMILITUDE.

Take any point E in a convenient position. Produce a line from A through E to K, making EK equal to two-thirds of AE (Prob. 10). Proceed in the same manner with each of the points

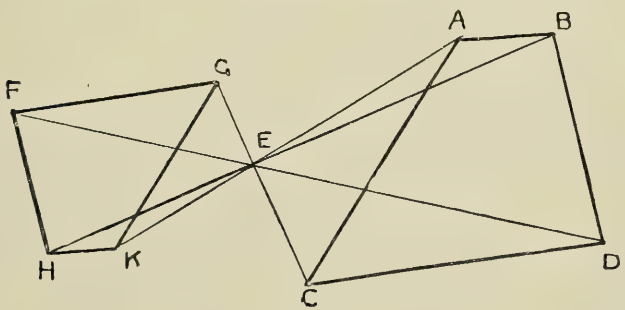


Fig. 276.

B, C, and D, which will give the points H, G, and F. Join KH, HF, FG, and GK. Then FGHK will be the trapezium required.

Problem 174.

To draw a circle one-half the radius of a given circle, by DIRECT *and* INVERSE *similitude.*

Let A be the centre of the given circle. Take any convenient point E. Bisect AE in B (Prob. 1). From E draw

tangents to the given circle (Prob. 84). Draw RB perpendicular to the tangent EO (Prob. 5). With B as centre, and BR as

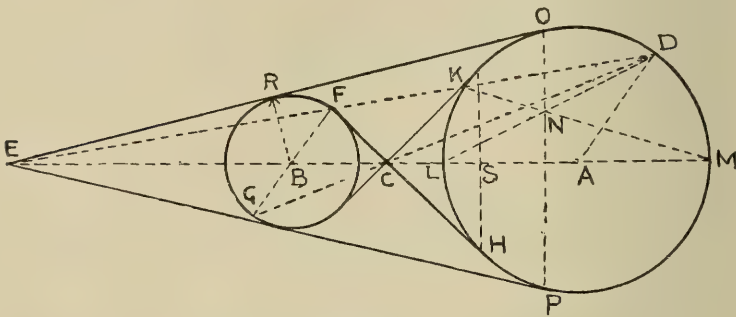


Fig. 277.

radius, draw the circle required. This would be by direct similitude.

To draw it by inverse similitude.

Take any convenient point C, and draw a line from A through C to B, making CB equal to one-half of AC. From C draw tangents to the given circle (Prob. 84) touching it in the points H and K. Draw any convenient radius AD. Through point B draw GF parallel to AD (Prob. 4). Produce a line from D through C till it meets GF in G. BG is the radius of the circle required.

Note.—C is called the internal, and E the external centre of similitude.

If we find the centres of similitude, we can easily draw either the *direct common tangents* or the *transverse common tangents* to two circles (see Probs. 97 and 98).

Let the two circles, with centres of similitude, in the illustration be given, and it is required to draw the two sets of tangents just described.

Join the centres A and B. Draw any convenient radius AD, and draw BF parallel to it (Prob. 4). Join DF. Produce BA to M, and draw a line from M to the point where DE cuts the circle. Join DL. Through the point of intersection N draw the

line OP perpendicular to AB (Prob. 5). These are the points of contact for the direct common tangents. Join OE and PE, two of the tangents required. Bisect CA in S (Prob. 1), and through S draw a line perpendicular to AB. Where this line cuts the circle are the points of contact for the transverse common tangents, which can be drawn through C.

Problem 175.

To draw an irregular polygon similar to a given polygon, but with sides two-thirds the length of the given polygon ABCDEFG.

It is not necessary that the centre of similitude should be taken outside the figure; if more convenient, we can use one of the angles of the figure, *e.g.* let A be the centre of similitude.

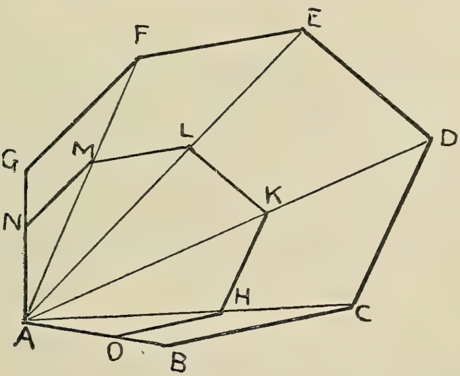


Fig. 278.

Draw lines from all the angles of the polygon to A. Make AK two-thirds of AD (Prob. 10), and divide all the other lines in the same proportion, which will give the points L, M, N, O, and H. Join NM, ML, LK, KH, and HO, which will give the polygon required.

Note.—The figures are similar, and the corresponding sides are parallel to each other.

Problem 176.

To draw an irregular pentagon similar to a given pentagon ABCDE, *but with sides one-half the length of the given pentagon, without using any centre of similitude.*

Draw FG in any convenient position parallel to AB, and

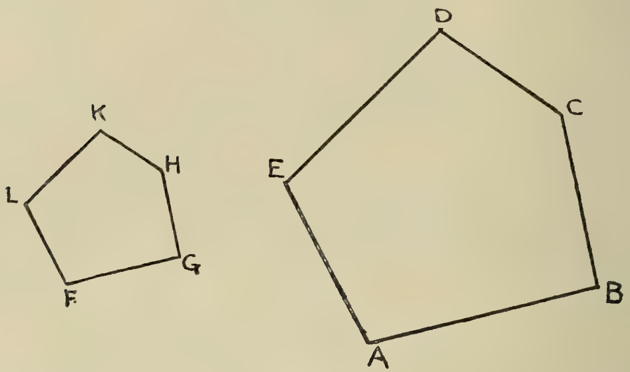


Fig. 279.

half its length (Prob. 4). From G draw GH parallel to BC and half its length. Proceed in the same way with the remaining sides, which will give the points K and L. Join HK, KL, and LF, which will give the pentagon required.

Problem 177.

To draw a curve or pattern similar to a given figure, but to two-thirds the scale.

Enclose the given figure in a convenient rectangular figure ABCD, and divide the sides of the rectangle into equal parts (Prob. 10). Join these divisions, which will divide the rectangle into a number of equal squares or rectangles. Draw another rectangle EFGH with sides two-thirds the length of the rectangle enclosing the given figure, and divide it in a similar manner.

Draw the curves to intersect these smaller divisions in the same places as the larger divisions are intersected by the given figure.

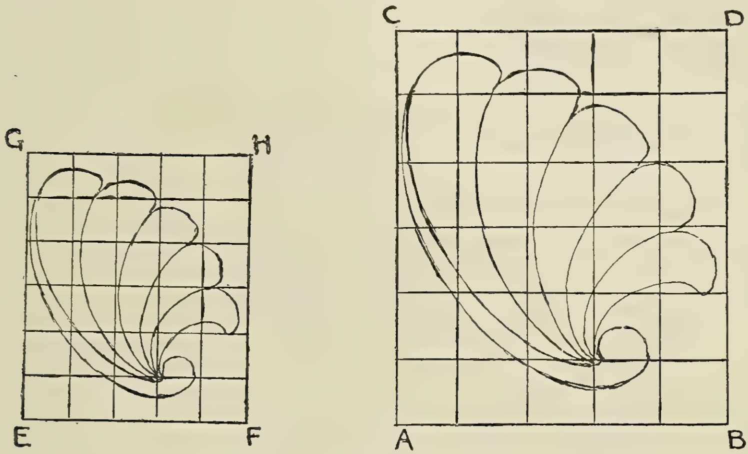


Fig. 280.

Note.—This method is used for enlarging or reducing maps or drawings to any scale when they are composed of curves or irregular lines.

CHAPTER XV

CONIC SECTIONS

A *conic section* is obtained by intersecting a cone by a plane. There are five different sections to a cone, viz.—

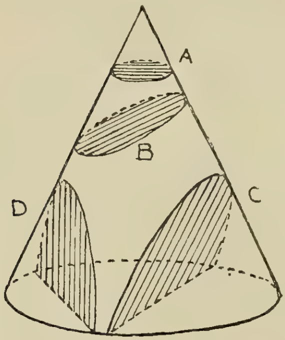


Fig. 281.

1. A *triangle*, when the plane cuts the cone through its axis.

2. A *circle*, when the plane cuts the cone parallel to its base, as at A, Fig. 281.

3. An *ellipse*, when the plane cuts the cone obliquely, without intersecting the base, as at B.

4. A *parabola*, when the plane cuts the cone parallel to one side, as at C.

5. An *hyperbola*, when the cone is cut by a plane that is perpendicular to its base, *i.e.* parallel to its axis, as at D, or inclined to the axis at a less angle than the side of the cone.

These curves can be drawn with the greatest accuracy and facility by the following arrangement. Cut a circular opening in a piece of thin card-board or stiff paper, and place it a short distance from a lighted candle; this will form a cone of light (Fig. 282). If we place a plane—or a piece of paper pinned to a drawing-board—so as to allow the light coming through the circular aperture to fall upon it, we can, by placing it in the several positions, intersect this cone of light so as to form

the required sections, which can then be traced. C is the candle, A is the circular aperture, and P the plane.

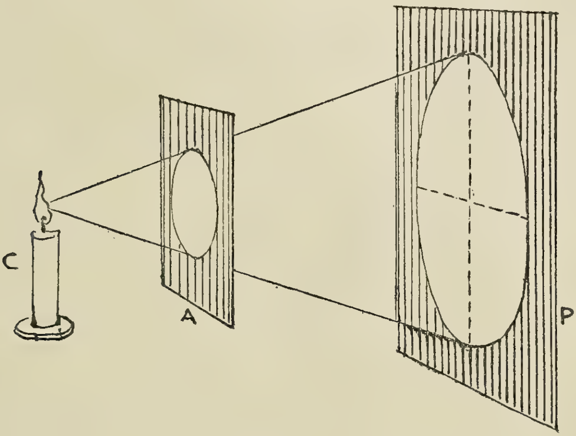


Fig. 282.

In Fig. 282 the plane is parallel to the aperture, so the section obtained is a *circle.*

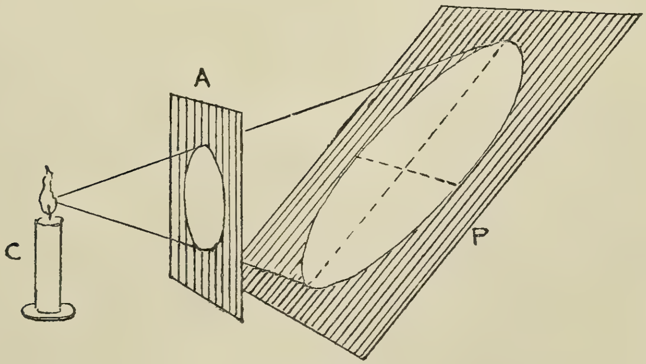


Fig. 283.

If the plane is placed obliquely to the aperture, as in Fig. 283, the section obtained is an *ellipse.*

By placing the plane parallel to the side of the cone, as in Fig. 284, the section gives the *parabola*.

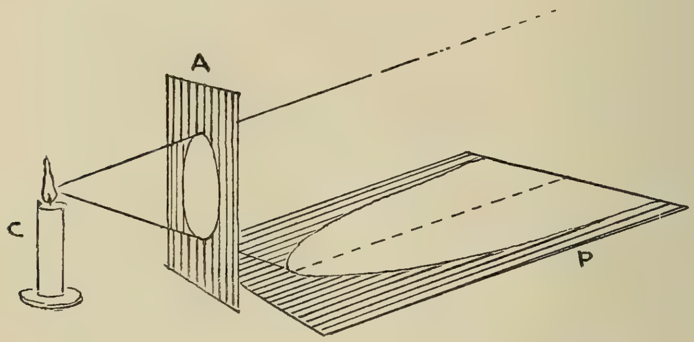


Fig. 284.

If we place the plane at right angles to the aperture, we obtain the *hyperbola*, Fig. 285.

By adjusting the positions of the candle, aperture, and plane,

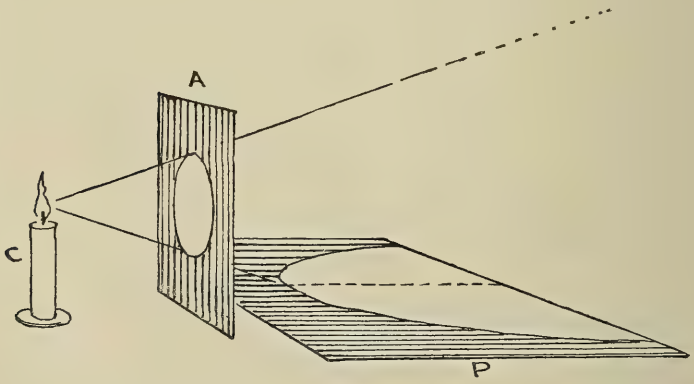


Fig. 285.

we can obtain a conic section to suit any required condition, both as to shape or size.

A *truncated cone* or *frustrum* is the part of the cone below section A, Fig. 281.

The Ellipse.

An ellipse has two unequal diameters or axes, which are at right angles to each other. The longer one is called the *transverse* diameter, and the shorter one the *conjugate* diameter.

The transverse diameter is also called the *major axis*, as AB (Fig. 286), and the conjugate diameter the *minor axis*, as CD.

Problem 178.

The two axes AB *and* CD *being given, to construct an ellipse.*

Take a strip of paper and set off upon it the distance FH, equal to half the major axis, and the distance FG, equal to half the minor axis. By keeping the point G on the major axis, and the point H on the minor axis, the point F will give a point in the ellipse. A succession of points can be found in this manner, through which draw a fair curve, which will give the required ellipse. An ellipse has two foci, as the points A and B, Fig. 287, and the sum of the radii from these two points is always equal.

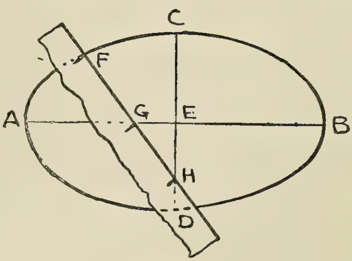


Fig. 286.

Let A and B represent two pins, and ABC a piece of thread. By placing the point of a pencil inside the thread at C, and keeping it drawn tight, the pencil on being moved would trace the ellipse. As the length of the thread is constant, the sum of the two radii must be constant also.

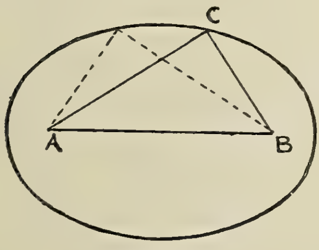


Fig. 287.

Problem 179.

The transverse and conjugate diameters of an ellipse being given, to find the foci.

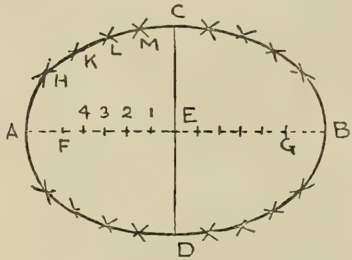


Fig. 288.

Let AB and CD be the two diameters bisecting each other at right angles in E. With C or D as centres, and radius equal to EA, set off on the transverse diameter the points F and G, which will be the foci required.

To construct an ellipse by means of intersecting arcs.

Divide the space between each focus and E into any number of parts. They need not be equal; in fact, it would be better if they were closer together as they approached the foci. From F and G as centres, with radii equal to A4 and B4, A3 and B3, A2 and B2, A1 and B1, describe arcs cutting each other on each side of the line AB in the points H, K, L, and M. Through these points draw a fair curve, which will give the ellipse required.

Problem 180.

To draw an approximate ellipse with arcs of circles, the major axis AB *and the minor axis* CD *being given.*

Let AB and CD intersect each other at right angles in E. From point A, with CD as radius, mark off the point F on AB, and divide FB into three equal parts (Prob. 10). From E, with two of these parts as radius, mark off on AB the points H and K. From H and K, with radius equal to HK,

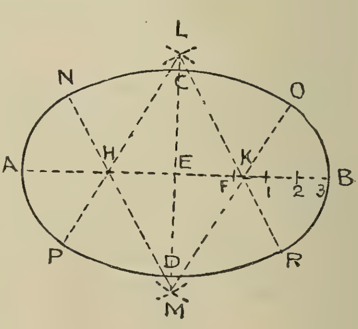


Fig. 289.

draw intersecting arcs in L and M. Produce lines through LH and LK, and through MH and MK. With H and K as centres, and radius equal to HA, draw the arcs PAN and OBR. With L and M as centres, and radius equal to MN, draw the arcs NCO and PDR, which will complete the required ellipse.

Problem 181.

To construct an ellipse by means of intersecting lines, the transverse diameter AB *and the conjugate diameter* CD *being given.*

Draw the lines AB and CD intersecting each other at right angles in the point E (Prob. 6). Draw HK and FG parallel to AB, and HF and KG parallel to CD (Prob. 4). Divide AH and BK into any number of equal parts, say four (Prob. 10), and AE and EB into the same number. Join C with the three points in AH and BK, and produce lines from D through the three points in AE and EB. Where these lines intersect those drawn from C, points in one-half of the ellipse will be obtained. Find corresponding points for the other half in the same manner, and draw a fair curve through the points obtained, which will give the required ellipse.

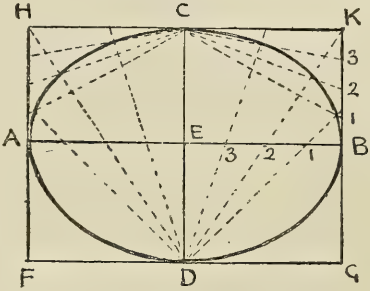


Fig. 290.

Problem 182.

Another method of drawing an ellipse by intersecting lines, the major and minor axes being given.

Let AB and CD be the major and minor axes; draw them intersecting each other at right angles in E (Prob. 6). With E as centre, and EA as radius, draw a circle; and with E as centre,

and EC as radius, draw another circle. Divide the quadrant FB into four equal parts in the points H, K, and L. Join these points to E, cutting the inner circle in the points M, N, and O. From the points H, K, and L draw lines parallel to FE (Prob. 4); and from M, N, and O draw lines parallel to EB till they meet the lines from H, K, and L. These will give the points for one-quarter of the ellipse. Find the points for the other three-quarters in the same manner, then draw a fair curve through all the points obtained, which will give the required ellipse.

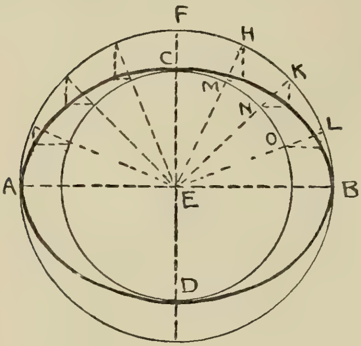


Fig. 291.

PROBLEM 183.

To find the normal and tangent to a given ellipse ABCD, *at a given point* P.

With C as centre, and radius equal to EA, draw the arc FH, which will give the two foci in F and H. Join the given point P with F and H, and bisect the angle FPH by the line PK (Prob. 13). PK is the normal. Draw the line NO through P, perpendicular to PK. This is the tangent required.

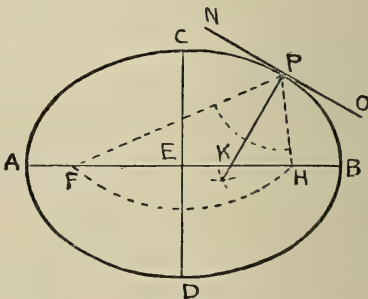


Fig. 292.

PROBLEM 184.

To complete an ellipse from an elliptical curve.

Let AB be the given curve. Draw any two sets of parallel

chords and bisect them (Prob. 1). Join the points of bisection in each set by lines meeting in C. Produce one of these lines till it meets the given curve in D. With C as centre, and CD as radius, set off on the given curve the point A. Join AD. Through C draw a line HK parallel to AD (Prob. 4), also the line CL perpendicular to AD (Prob. 6). Produce CL to M,

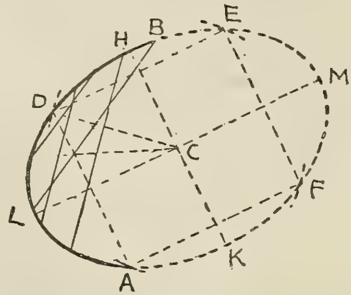


Fig. 293.

making CM equal to CL. Also make CK equal to CH. Then LM will be the transverse and HK the conjugate diameters. The ellipse can then be completed by any of the constructions already described. If from A and D lines are drawn parallel to LM, and with C as centre, and radius CD, arcs are drawn intersecting these lines in E and F, these will be two more points in the ellipse.

Problem 185.

To draw an ellipse to pass through three given points A, B, C.

Join AC and bisect it in D. Join BD. From A and C draw

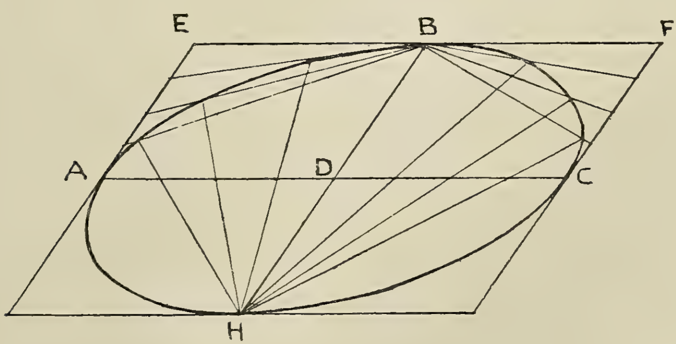


Fig. 294.

the lines AE and CF parallel to BD. Through B draw the line

EF parallel to AC. Produce BD to H, and make DH equal to BD. Divide AD and DC into any number of equal parts, say four, and also divide AE and CF into a corresponding number of equal parts. Join the divisions on AE and CF to B. From H, through the divisions on AC, draw lines till they meet the corresponding lines drawn to B. Draw a fair curve through these points, which will give half of the required ellipse. Proceed in the same way with the other half.

Problem 186.

To determine points for drawing a parabola, the focus A *and the directrix* BC *being given.*

Draw the line EAD perpendicular to the directrix BC (Prob. 6), which will give the axis. Bisect AD in F (Prob. 1), which will be the vertex of the curve. Take any points a, b, c, d, and e in the axis, and draw perpendiculars through them. From A as centre, mark off on the perpendiculars, arcs with radii equal to aD, bD, cD, dD, and eD, cutting the perpendiculars in a', b', c', d', and e'. These are the points required for the lower half of the parabola. The points above the axis are found in the same manner.

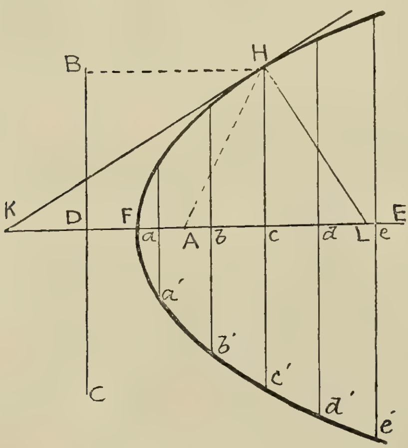


Fig. 295.

To draw a tangent to a parabola at a given point H.

Join AH. From A set off AK on the axis produced equal to AH. Join KH, which will be the required tangent. This could also be found by drawing a line from H parallel to the axis till it meets the directrix in B, and then bisecting the angle

AHB by the line KH (Prob. 13), which is the tangent. If from H we draw the line HL perpendicular to the tangent, it will be the normal.

PROBLEM 187.

To draw a parabola, an abscissa AB *and an ordinate* BC *being given.*

Complete the rectangle ABCD. Divide BC into any number of equal parts, say six (Prob. 10), and CD into the same number. From each division in BC draw lines parallel to CD (Prob. 4), and from each of the divisions in CD draw lines to the vertex A. Where these lines of corresponding numbers intersect, *e.g.* where 1 intersects with 1′, 2 with 2′, etc., are points in the parabola. Find corresponding points on the opposite side of the axis, and draw a fair curve through them.

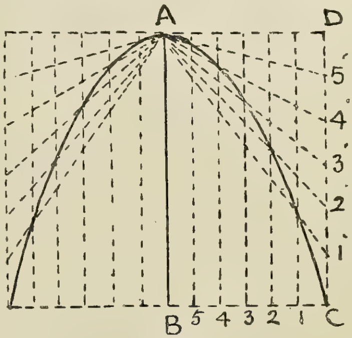


Fig. 296.

PROBLEM 188.

To draw an hyperbola, the diameter AB, *an ordinate* CD, *and an abscissa* BD *being given.*

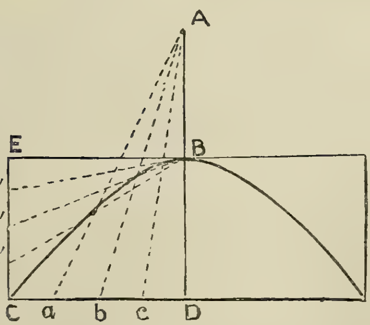


Fig. 297.

Draw BE parallel to CD (Prob. 8), and complete the rectangle. Produce BD, and make AB equal to the given diameter. Divide CD and CE into any number of equal parts, say four (Prob. 10), *a*, *b*, *c*. The divisions on CD join to A, and those on CE to B. The intersection

of the corresponding lines, *e.g.* where a intersects a', b b', and c c', are points in the hyperbola required. Find corresponding points for the other half, and draw a fair curve through them.

Problem 189.

A form of hyperbola frequently used by engineers is that shown in the following figure, and is called a *rectangular hyperbola.* Let AB and AC represent two axes, and E the vertex of the curve. Complete the rectangle ABDC. Take any point H in CD and join it to A. Let fall a perpendicular from E

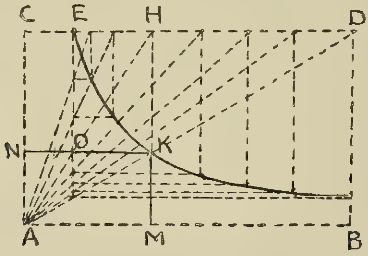


Fig. 298.

till it meets HA in O (Prob. 6). From O draw OK parallel to AB till it meets a line from H parallel to AC in the point K (Prob. 4). This will be one point in the curve, and any number of other points may be found by taking fresh points on CD and treating them in a similar manner.

The peculiar property of this figure is that, if we take any point in the curve and draw lines from it perpendicular to the lines AB and AC,—for example, KN and KM,—the rectangle contained by the two lines is always equal, *i.e.* KN × KM would be the same for any point in the curve.

Problem 190.

A mechanical method of drawing a parabola or hyperbola.

Let AB represent the edge of a drawing-board and CD the

edge of a tee-square. Take a piece of string equal in length to CD, fix one end at D and the other at E, which is the focus of the curve. If a pencil be held against the string, and kept tight against the tee-square, it will trace half a parabola on moving the tee-square upwards. AB is the directrix, and K the vertex of the curve. Compare this method with the construction of Prob. 186.

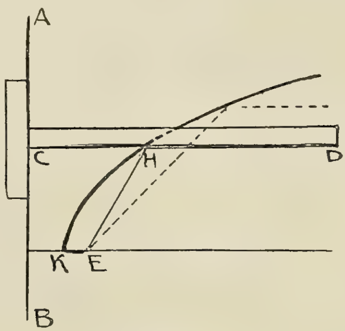


Fig. 299.

If the angle DCA were an acute or obtuse angle instead of a right angle, the pencil would trace an hyperbola.

Problem 191.

To draw an oval by arcs of circles, its transverse diameter AB *and its conjugate* CD *being given.*

Set off on AB the distance AE equal to half the conjugate diameter. Through E draw the line FG perpendicular to AB (Prob. 5). With E as centre, and EA as radius, draw the semicircle CAD. From C and D set off CF and DG equal to EA. From B set off BH equal to half of EA. From F and G draw lines through H. With F and G as centres, and FD as radius, draw the arcs DL and CK. With H as centre, and HK as radius, draw the arc KBL, which will complete the oval required.

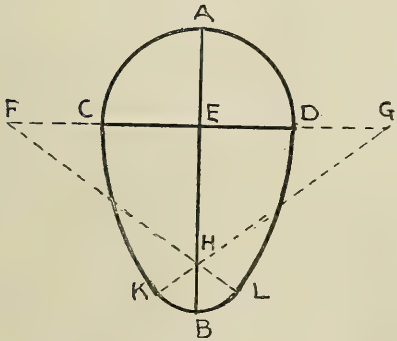


Fig. 300.

The curves formed by the conic sections are of the utmost

importance both in science and art: they are traced in space by the heavenly bodies in their courses; they are used by engineers where great strength is required, such as the construction of bridges; and they form the contour of mouldings, etc. They are also among the most beautiful of art forms: those subtle curves that we admire so much in the outline of Japanese hand-screens and vases are modifications of them.

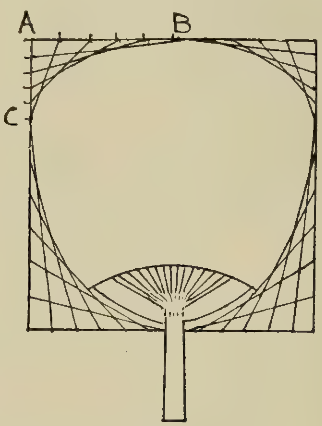


Fig. 301.

Fig. 301 is an illustration showing how these curves are applied to art forms.

To draw the curve CB. Draw the lines AB and AC at right angles to each other (Prob. 8). Divide each into the same number of equal parts (Prob. 10), and join them. Proceed in the same manner with the other curves.

CHAPTER XVI

CYCLOIDAL CURVES, ETC.

If a circle is rolled along a line in the same plane, a point in the circle will describe a curve of a class called *cycloidal.*

The line along which the circle rolls is called a *director*, and the point itself is called the *generator.*

The curve is called a Cycloid when the generator point is in the circumference of the rolling circle and the director is a straight line, but a Trochoid when the point is not in the circumference of the circle.

When the director is not a straight line, but the outside of another circle, and the generator is in the circumference of the rolling circle, the curve described is called an Epicycloid ; but when the point or generator is not in the circumference of the rolling circle, it is called an Epitrochoid.

If the director is the inside of a circle and the generator a point in the circumference of the rolling circle, the curve is called a Hypocycloid ; but if the generator is not in the circumference of the rolling circle, it is called a Hypotrochoid.

In constructing a cycloid it is necessary to make a line equal in length to the arc of a semicircle. The exact relation between the diameter and circumference of a circle cannot be definitely fixed, but the following problem will enable us to arrive at an approximation, the error of which is only about $\frac{1}{100,000}$.

Problem 192.

To draw a line equal to the length of a semicircle.

Let AC represent the radius. Draw the semicircle ABD. Produce AC to D, and draw BC perpendicular to it. From A and D draw tangents parallel to BC, and through B draw a tangent parallel to AD. From B set off BE equal to the radius, and draw the line BF through E. Produce the tangent through

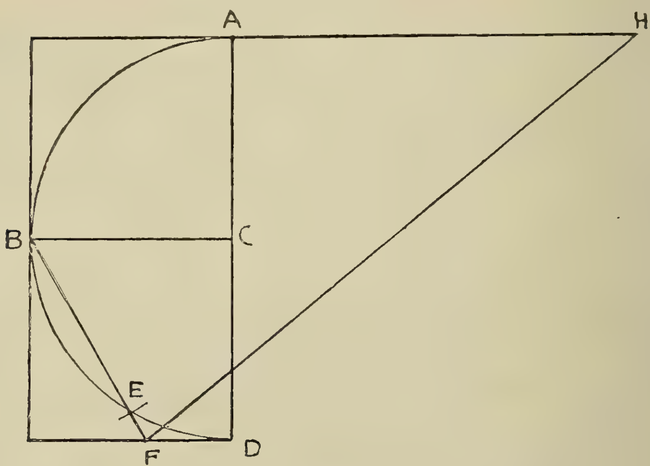


Fig. 302.

A to H, and make AH equal to AD. Join HF, which will be the line required.

If we take AC to represent a length equal to the diameter of a circle, then HF will equal the circumference.

We are also enabled to find the length of an arc by this means; *e.g.* the arc to the chord formed by one side of a pentagon. If FH is equal to the circumference, then $\frac{FH}{5}$ = the length of the arc required.

Problem 193.

To draw a cycloid.

Let AB be the director, and *p* the generator or point in the rolling circle AM*p*. Draw AB equal in length to half the circumference of the circle AM*p*, and divide it into any number of equal divisions (Prob. 10), say six, *d*, *e*, *f*, *g*, *h*, and *k*. Divide the semicircle into the same number of equal divisions (Prob. 14), and draw lines from each division parallel to the director AB (Prob. 4). Draw the line CK from the centre of the circle parallel to AB. Draw lines perpendicular to AB (Prob. 5) at

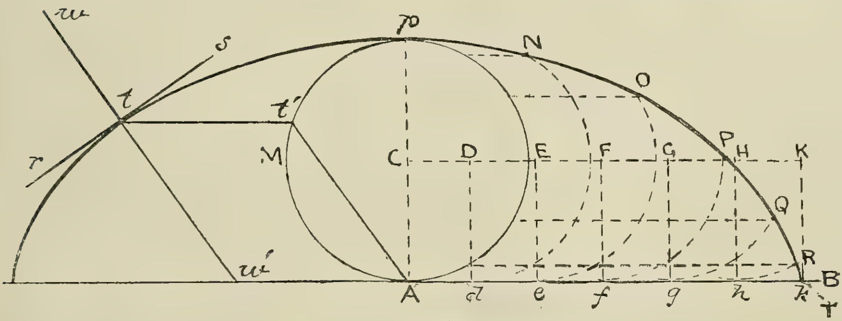


Fig. 303.

the points *d*, *e*, *f*, *g*, *h*, and *k* till they meet the line CK in the points D, E, F, G, H, and K. With each of the points D, E, F, G, H, and K as centres, and a radius equal to C*p*, draw arcs cutting the parallel lines drawn from the divisions in the semicircle in the points N, O, P, Q, R, and T. This will give points in half the cycloid. Find the corresponding points for the remaining half, and draw a fair curve through the points, which will give the cycloid required.

To determine the tangent and normal to the curve at any point *t*. Draw the line *tt'* parallel to AB till it meets the generating circle at *t'*. Join *t'*A. Through *t* draw the line *ww'* parallel to *t'*A. This will be the normal to the curve. The tangent *rs* is at right angles to this line.

Problem 194.

To draw a trochoid.

Let AB be the director, and p the generator outside the rolling circle D, but to revolve with it. Divide the circle D into any number of equal parts, say twelve (Prob. 14). Through the centre of the circle draw a line parallel to AB (Prob. 4), and from C, along this line, set off eight divisions N, O, P, R, S, T, V, and W, each equal to one of the divisions on the circle D.

With C as centre, and radius Cp, draw the circle E, and

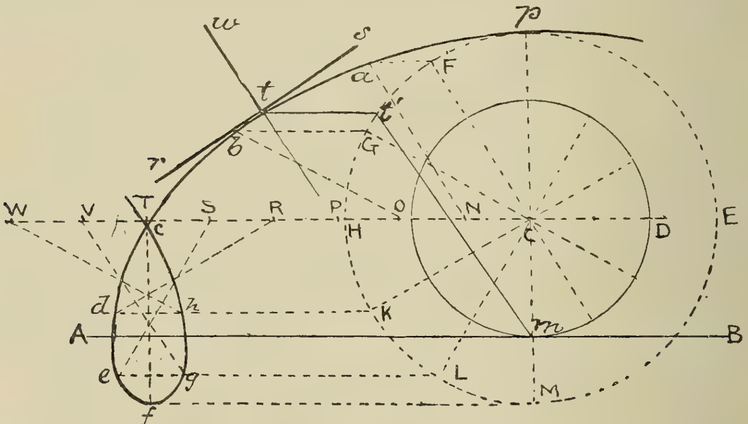


Fig. 304.

produce the divisions in circle D till they meet it in the points F, G, H, K, L, and M, and from these points draw lines parallel to AB.

With the points C, N, O, P, R, S, T, V, and W, and a radius equal to Cp, set off the points a, b, c, d, e, f, g, and h on the parallels drawn from the points F, G, H, K, L, and M. Draw a fair curve through these points.

Note.—Half of the trochoid would be from p to f, but the curve is continued to show the loop which is characteristic of trochoid curves when the generator is outside the rolling circle.

To draw a tangent and normal at any point t. Draw the

line tt' parallel to AB till it meets the circle E in t'. Join $t'm$. Draw the normal wt parallel to $t'm$, and the tangent rs at right angles to it.

Problem 195.

To draw an epicycloid.

Note.—The length of the arc of *director* is to the whole circle as the radius of the rolling circle is to the radius of the *director*; *e.g.* if radius of rolling circle = 1 inch, and that of director = 6 inches; then the director = $\frac{1}{6}$ of a circle.

Let AB be the director, which is a part of a circle, and p

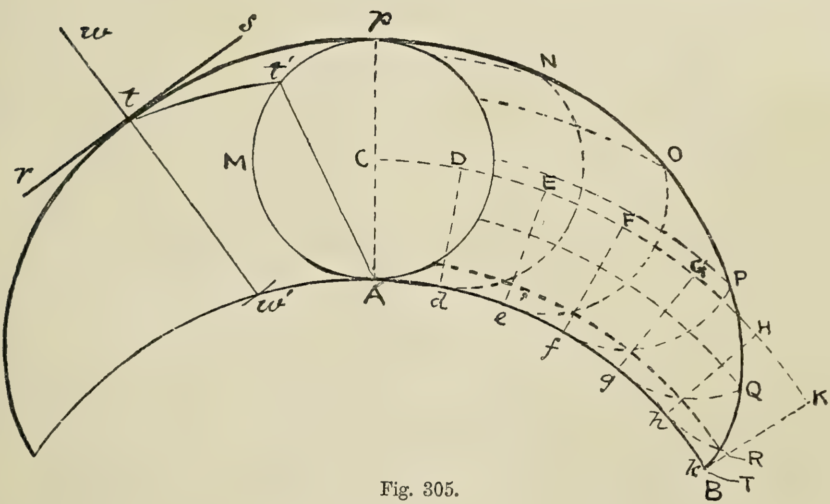


Fig. 305.

the generator. Make AB equal in length to half the rolling circle AMp, and divide it into any number of equal divisions, say six, d, e, f, g, h, and k (Prob. 14). Divide the semicircle into the same number of equal divisions, and draw lines from these points, as well as from the centre of the circle, concentric with the arc AB. From the centre of the circle that contains the arc AB draw lines through the points d, e, f, g, h, and k till they meet the arc drawn from the centre of the rolling circle. With D, E, F, G, H, and K as centres, and a radius equal to

Cp, draw arcs till they meet the concentric arcs drawn from the divisions of the semicircle in the points N, O, P, Q, R, and T. Find corresponding points on the opposite side, and draw a fair curve through all the points, which will be the epicycloid required.

At any point t to draw a tangent and normal to the curve, proceed as follows. Draw the arc tt' concentric to the arc AB till it meets the generating circle in t'. Join t'A. With t as centre, and radius equal to t'A, draw an arc intersecting AB at w'. Join tw', and produce it to w. This is the normal. The tangent rs is at right angles to it.

Problem 196.

To draw an epitrochoid.

Let AB be the director, which is the arc of a circle, and p the

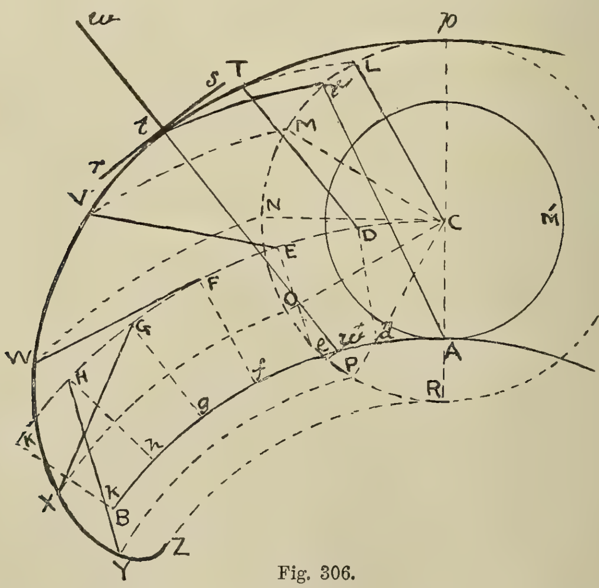


Fig. 306.

generator outside the circumference of the rolling circle M′.

Make AB equal in length to half the circumference of the circle M, and divide it into any number of equal parts, say six (Prob. 14). Divide half of the circle M′ into the same number of equal parts. With C as centre, and radius C*p*, draw the circle *p*SR, and draw lines from C through the divisions in the semicircle till they meet this circle in the points L, M, N, O, P, and R. Through each of these points, as well as the centre C, draw arcs concentric with the director AB. From the centre of the circle containing the arc AB draw radiating lines till they meet the arc drawn from C in the points D, E, F, G, H, and K. With each of these points as centres, and a radius equal to C*p*, set off the points T, V, W, X, Y, and Z on the arcs drawn from the divisions of the circle *p*SR. Draw a fair curve through these points, which will give half of the epitrochoid required.

The tangent and normal at any point *t* are determined as follows. Draw the arc *tt′* concentric to the arc BA till it meets the circle *p*SR in *t′*. Join *t′*A. With *t* as centre, and radius equal to *t′*A, draw an arc cutting BA in *w′*. Join *tw′*, and produce the line to *w*. This will give the normal. The tangent *rs* is drawn at right angles to it.

Problem 197.

To draw a hypocycloid.

Let AB be the director, which is the arc of a circle, and *p* the generator, which is a point in the circumference of the rolling circle M. Make AB equal in length to half of the circle M, and divide it into any number of equal parts, say six, *d, e, f, g, h,* and *k* (Prob. 14). Divide the semicircle into the same number of equal parts, and from the centre of the circle containing the arc AB draw concentric arcs from these points, as well as from the centre C. Draw lines from the points *d, e, f, g, h,* and *k* towards the centre of the circle containing the arc AB till they meet the arc from the centre C in the points D, E, F, G, H, and K. With each of these points as centres, and a radius equal to C*p*, draw arcs till they meet the concentric

arcs drawn from the divisions of the semicircle in the points N, O, P, Q, R, and T. Find the corresponding points for the

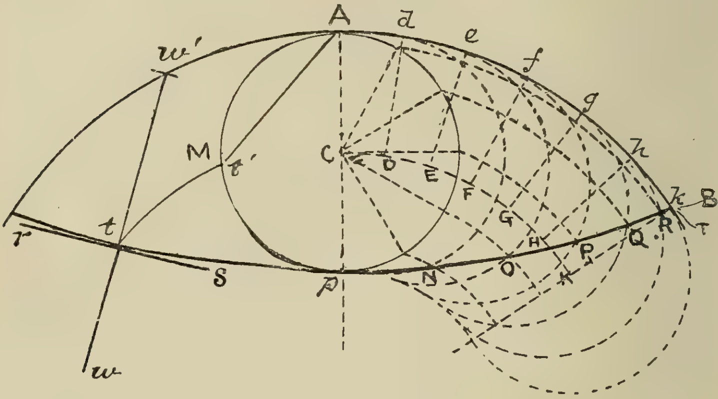


Fig. 307.

other half, and draw a fair curve through all the points, which will give the hypocycloid required.

The tangent and normal at any point t are thus obtained. Draw the arc tt' concentric to the arc AB till it meets the generating circle at t'. Join t'A. With t as centre, and radius equal to t'A, set off on AB the point w'. Join tw', and produce the line to w. This is the normal. The tangent rs is at right angles to it.

Problem 198.

To draw a hypotrochoid.

Let AB be the director, which is the arc of a circle, and p the generator, which is a point outside the rolling circle M, but made to revolve with it. Make AB equal in length to half the circle M, and divide it into any number of equal parts, say six, d, e, f, g, h, and k (Prob. 166). Divide half of the circle M into the same number of equal parts. With C as centre, and radius Cp, draw the circle S, and draw lines from the centre C through

the divisions in the semicircle till they meet the circle S in the points l, m, n, o, p, and r.

From the centre of the circle containing the arc AB draw concentric arcs from these points, as well as from the centre C.

Draw lines from the points d, e, f, g, h, and k towards the centre of the circle containing the arc AB, till they meet the arc drawn from C in the points D, E, F, G, H, and K. With each of these points as centres, and a radius equal to Cp, set off

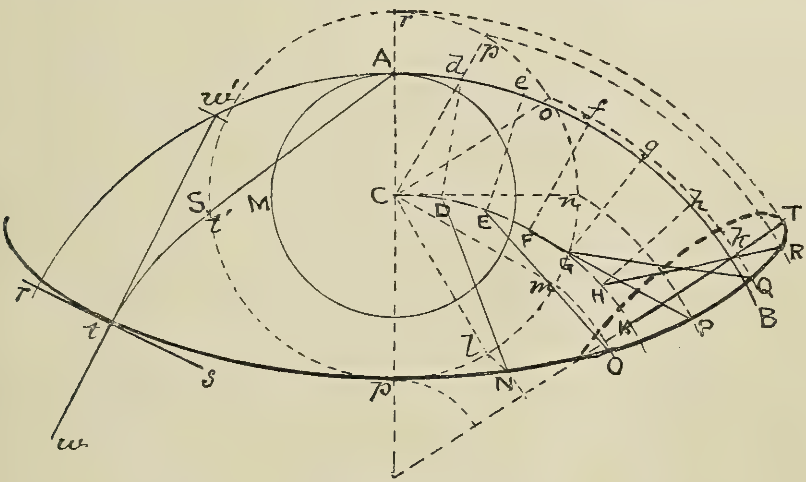


Fig. 308.

the points N, O, P, Q, R, and T on the concentric arcs drawn from the divisions in the circumference of the circle S.

Draw a fair curve through these points, which will give half of the hypotrochoid required.

Note.—The thick dotted line shows the completion of the loop.

To determine the tangent and normal at any point t, proceed as follows. Draw the arc tt' concentric to AB till it meets the circle S in t'. Join t'A. With t as centre, and radius equal to t'A, set off on AB the point w'. Join tw', and produce the line to w. This is the normal. The tangent rs is at right angles to it.

The hypocycloid possesses a very peculiar and important property, which is illustrated in Fig. 309. If the diameter Ap of the circle containing the generator p is half of the diameter of the circle containing the director AB, the generator p will describe the straight line pB instead of a curve. While the point

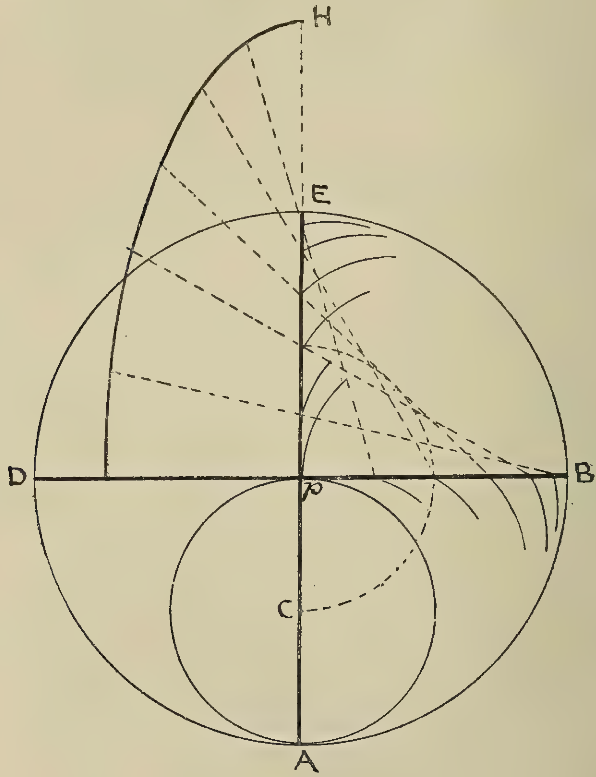


Fig. 309.

p is describing the straight line pB, the point A will describe the straight line Ap.

If the generating circle were to make a complete revolution round the directing circle, the points A and p would describe the diameters DB and AE to the larger circle, at right angles to each other.

If a point H be taken anywhere on the diameter AE, either inside or outside of the directing circle, this point will describe an ellipse, one-quarter of which is shown. This principle has been used in drawing the ellipse in Fig. 286. So both the ellipse and a straight line belong to the cycloidal series.

Problem 199.

To draw the involute of a circle.

The involute is a particular kind of epicycloidal curve. If a flexible line is unwound from a circle or curve, and at the same time kept straight, any point in this line will describe a curve

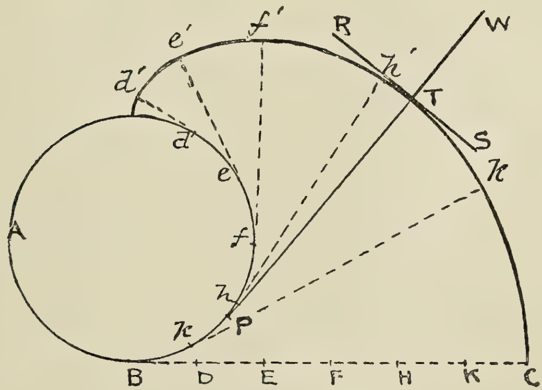


Fig. 310.

called the *involute* of the circle or curve. The generating circle or curve is called its *evolute.*

Let A be a given circle, Fig. 310. From B draw the tangent BC (Prob. 84) equal in length to half the circumference of the given circle. Divide this tangent into any number of equal parts, say six (Prob. 10), D, E, F, H, and K, and divide half of the circumference of the given circle into the same number of equal parts d, e, f, h, and k. From each of these points draw tangents to the semicircle. Make $dd' = \text{BD}$, $ee' = \text{BE}$, $ff' = \text{BF}$, $hh' =$

BH, and $kk' = $ BK. Through the points thus found draw a fair curve, which will be the involute required.

The tangent and normal at any point T are obtained thus. Draw the line TP tangential to the circle A, and produce it to W. This is the normal; the tangent RS is at right angles to it.

Problem 200.

To construct a continuous curve, by a combination of arcs of different radii, through a number of given points A, B, C, D, E, F, G, and H.

Join the points AC, CD, DE, etc. Find the centre K of the circle containing the arc ABC (Prob. 34). Join CK.

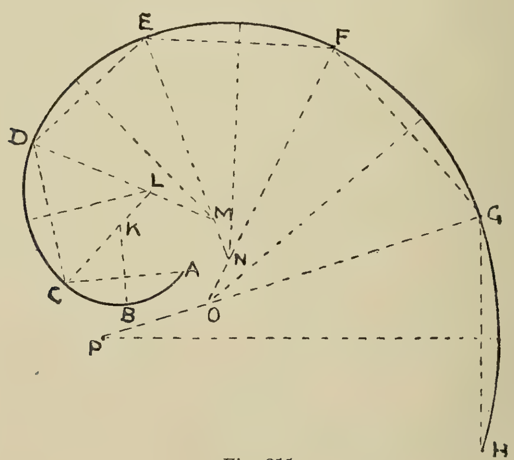


Fig. 311.

Bisect the line CD (Prob. 1), and produce the bisecting perpendicular till it meets CK produced in L. Join DL. Bisect the line DE, and produce the bisecting perpendicular till it meets DL produced in M. Find the remaining points N, O, and P in the same manner. The points K, L, M, N, O, and P are the centres of the circles containing the arcs necessary for joining the given points.

Problem 201.

To draw a common spiral of five revolutions on a given diameter AB *by means of semicircles.*

Divide AB into ten equal parts (Prob. 10). Bisect the sixth division from A in C (Prob. 1). With C as centre, and CK as radius, draw the semicircle KL. With K as centre, and radius KL, draw the semicircle LH. With C as centre, and radii CH, CF, CE, and CD, draw all the semicircles above the line AB; and with K as centre, and radii KF, KE, KD, and KA, draw all the semicircles below the line AB, which will complete the required spiral.

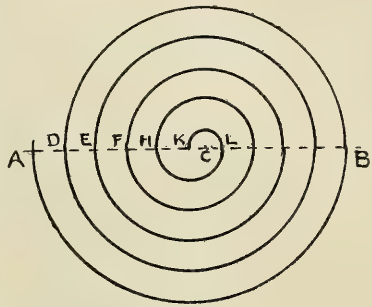


Fig. 312.

Problem 202.

To construct an Archimedean spiral of one revolution.

Draw a circle and divide it by radii into any number of

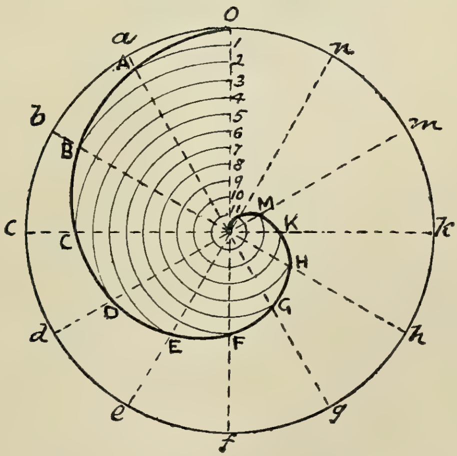


Fig. 313.

equal parts, say twelve (Prob. 14), *a*, *b*, *c*, *d*, etc. Divide the radius *o* into a corresponding number of equal parts 1, 2, 3, 4, etc. (Prob. 10). From the centre of the circle, with radius 1, draw an arc till it meets the radius *a* in A, and from 2 till it meets the radius *b* in B, and so on till the whole twelve are completed. Draw a fair curve through these points, A, B, C, D, etc., which will give the spiral required.

PROBLEM 203.

To construct an Archimedean spiral of two revolutions.

Draw a circle and divide it into equal parts by radii, as

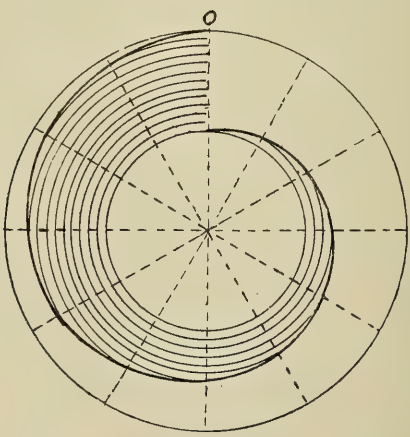


Fig. 314.

described in the preceding problem. From the centre of the circle, with half its radius, draw another circle. The part of the spiral contained in this inner circle, *i.e.* one revolution, will be drawn precisely the same as the one described in the preceding problem.

Divide the outer half of radius *o* into twelve equal divisions (Prob. 10), and describe arcs from the centre of the circle, as before described, and draw the curve through these points, which will complete the spiral required.

LOGARITHMIC SPIRALS.

The *logarithmic spiral* was discovered by Descartes. It is also called the equiangular spiral, because the angle the curve makes with the radius vector is constant. The curve also bears a constant proportion to the length of the radius vector.

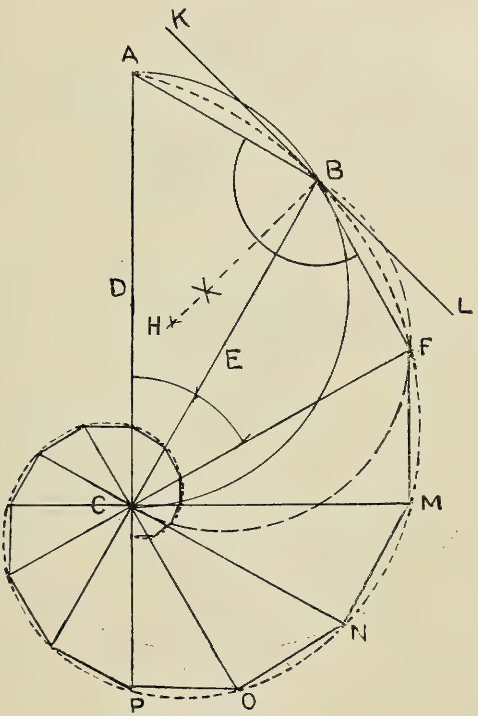


Fig. 315.

Take any line AC for the radius vector, and bisect it in D (Fig. 315). With D as centre, and radius DA, draw the semicircle ABC. From the points A and C draw any two lines AB and CB cutting the semicircle. Then ABC is a right-angled triangle.

Bisect the line BC in E. With E as centre, and EB as radius, draw the semicircle BFC. Make the angle BCF equal to

the angle ACB, and produce the line till it meets the semicircle in F. Join BF. The triangle BFC is then similar to the triangle ABC. By repeating this construction we obtain a succession of similar triangles radiating from a common centre C, and all forming equal angles at this point. The exterior points of these triangles, viz. A, B, F, M, N, O, etc., are points in the required spiral.

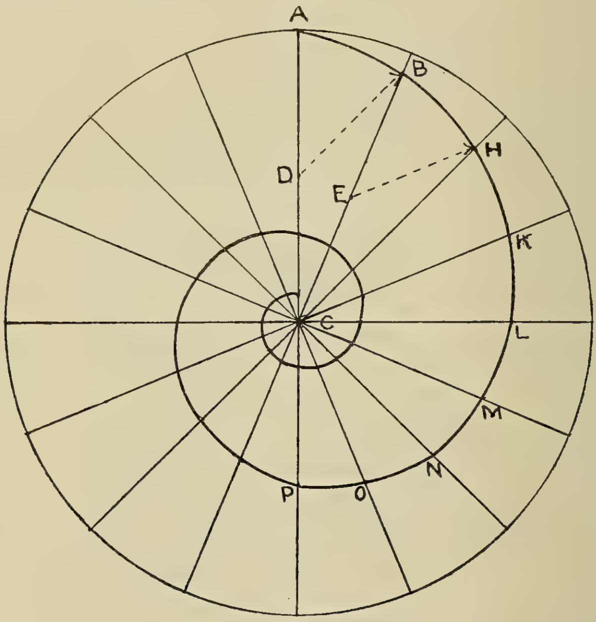


Fig. 316.

As each triangle with its curve forms a similar figure, it is evident that the curve must form a constant angle with its radius vector, *i.e.* the line radiating from C, and the portion of the curve accompanying each triangle, must also bear a constant proportion to the length of its radius vector.

If we bisect the angle ABF by the dotted line HB, this line will be the normal to the curve; and the line KL, being drawn at right angles to HB, is the tangent to the spiral.

As all the angles at C are equal, the spiral could be con-

structed with greater facility by first drawing a circle and dividing it into an equal number of parts by radii, as shown in Fig. 316.

Let AC be the radius vector. Bisect it in D. With D as centre, and radius DA, draw an arc cutting the next radius CB in B. Proceed in the same manner with each radius in succession, which will determine the points H, K, L, M, N, O, P, etc. Draw a fair curve through these points, and we shall obtain a logarithmic spiral.

The greater the number of radii used in the construction, the larger will be the angle BAC; but the angle ABC will always be a right angle, as will be seen by the construction in Fig. 315.

PROBLEM 204.

To draw a spiral adapted for the Ionic volute by means of arcs.

Divide the given height AB into eight equal parts (Prob.

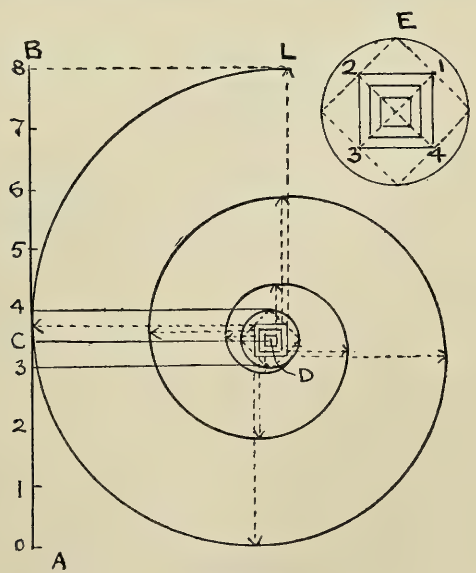


Fig. 317.

10). Bisect the fourth part in the point C (Prob. 1), and from it draw a line perpendicular to AB (Prob. 5). Make this line equal in length to four of the divisions of AB, which will give the eye of the volute D. This is shown to a larger scale at E. With D as centre, draw a circle with a radius equal to C4. Inscribe a square in this circle (Prob. 56), and bisect each of its sides in the points 1, 2, 3, and 4 (Prob. 1). Join these points, and draw diagonals. Divide each semi-diagonal into three equal parts and join them (Prob. 10), thus making three complete squares parallel to each other. The corner of each of these squares in succession will be the centre of one of the arcs, commencing at 1, with L as radius, as shown by dotted lines and arrow-heads.

Diagram showing special forms of eccentrics or cams.

The half of the figure drawn to the left of the line AE shows

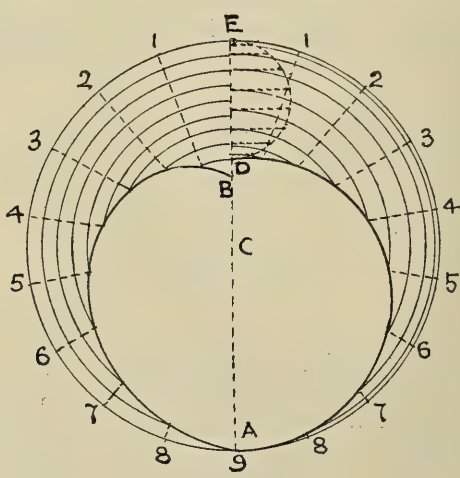


Fig. 318.

the form of a cam that will produce a rectilinear motion that is equable throughout its length.

Let BE represent the amount of rectilinear motion required.

One-half of BE will give the shortest radius of the cam, and one and a half times BE the longest radius. Draw a circle with a radius equal to one and a half times BE, and divide it into any number of equal parts, say eighteen, and draw radii. Divide BE into half this number, *i.e.* nine (Prob. 10), and draw concentric arcs, with C as centre, stopping each arc against the radii in succession. Draw a fair curve through the points thus obtained. This will give half of the cam.

The form of the half cam to the right of the line AE is one in which the radius varies by a law more easily understood from the construction than by any description (Prob. 152). DE represents the amount of traverse. Bisect this line, and draw a semicircle upon it. Divide this semicircle into the same number of equal divisions as are contained in half the large circle, *i.e.* nine, and from these divisions draw lines perpendicular to DE (Prob. 5). From the bases of these perpendiculars draw concentric arcs from C till they meet the radii in succession, and join the points thus obtained by a fair curve. The rectilinear motion produced by this cam is not equable.

Problem 205.

To draw a conchoid.

The conchoid is a curve invented by Nicomedes in the fifth century. It has the peculiar property of always approaching nearer a straight line as it is produced, but would never meet it.

Draw the line AB, called the *asymptote*, and the line CF perpendicular to it (Prob. 5). On CF select the point P, which is called the *pole*.

From C, along AB, set off any number of points at pleasure, 1, 2, 3, etc., and draw lines from P through these points. Produce the perpendicular CF below C, and select the point D. CD is called the *diameter*.

Set off along the lines radiating from P distances equal to CD, as 1H, 2K, etc., and draw a curve through these points. This

is called the *inferior* conchoid. If we set off these distances above the line AB, as $1h$, $2k$, etc., and draw a curve through

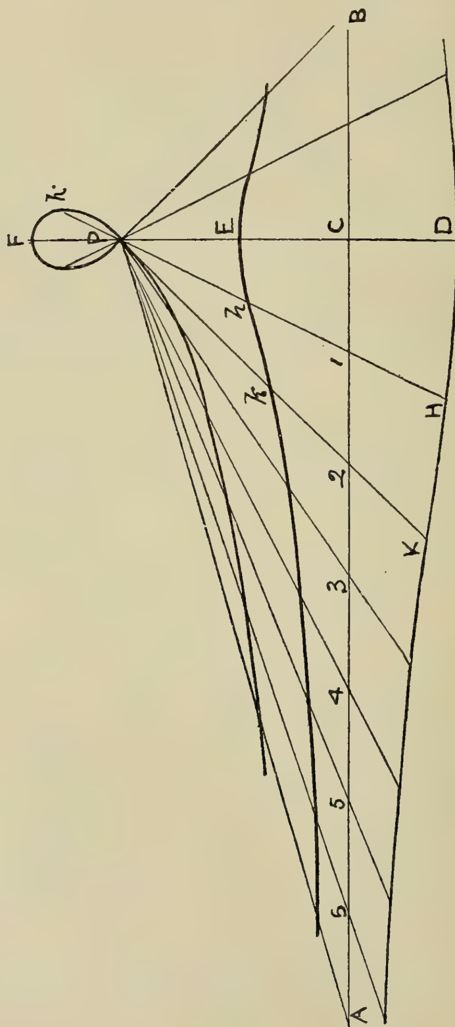


Fig. 319.

these points, it would be called the *superior* conchoid. If we take the diameter greater than the length CP, the curve would have a loop.

Problem 206.

To draw a cissoid.

The cissoid was invented by Diocles in the second century.

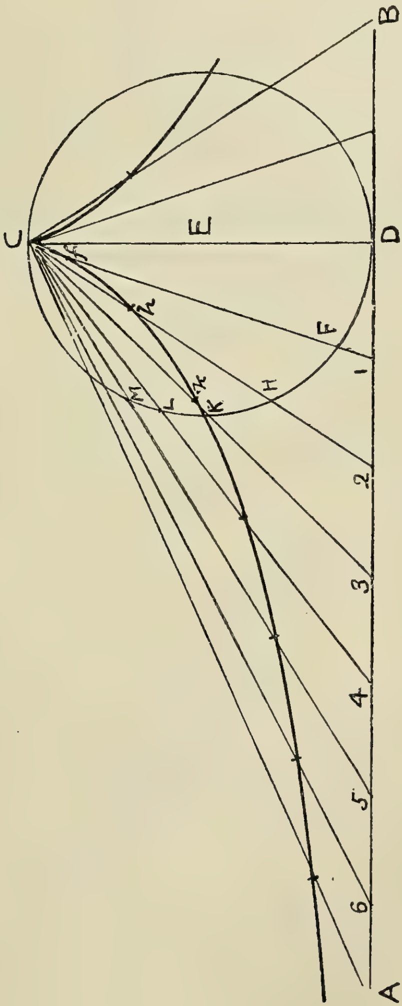


Fig. 320.

Take any line AB, and draw CD perpendicular to it (Prob. 5). Bisect CD in E (Prob. 1). With E as centre, and EC as radius,

draw a circle. Select any points on AB at pleasure, as 1, 2, 3, etc., and join them to C, cutting the circle in the points F, H, K, etc. Take the lengths 1F, 2H, 3K, etc., and set them off from C in *f*, *h*, *k*, etc. Draw a curve through these points, which will be the required cissoid.

This curve occurs in the mechanism called link-motion.

The *catenary* curve is formed by a chain suspended between its ends, which are fixed. A similar curve is found by rolling a parabola and fixing a point at its focus, which will trace a curve called a *centenoid*.

A point fixed at the focus of a rolling hyperbola would trace a curve called a *nonoid*.

Problem 207.

To draw an ogee or cyma reversa.

Note.—The *cyma recta* has the hollow part of the moulding at the top.

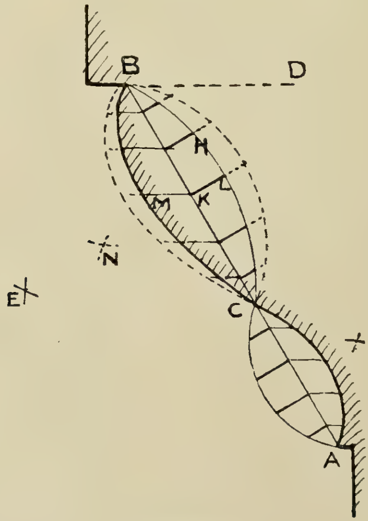


Fig. 321.

Let A and B be the extreme points of the moulding. Join AB,

and take any point C. With B and C as centres, and with any radius, draw the intersecting arcs at E. With E as centre, and EB as radius, draw the arc BHC. Divide BC into any number of parts. These divisions should be closer together towards the ends of the curve. On these divisions draw lines perpendicular to BC till they meet the arc BHC. From these same divisions on BC draw lines parallel to the base DB. Take the lengths of the perpendiculars and set them off on the horizontal lines, *e.g.* make KM equal to KL, etc. Treat the line AC in the same manner. Draw a fair curve through the points thus found.

Note.—The curve of the moulding will depend upon the radius of the intersecting arcs at E; *e.g.* if we take a smaller radius, BN for instance, the curve will be sharper, as shown by the dotted line.

A very important application of the principle of Practical Geometry is to ascertain the several movements of the various portions of a piece of mechanism; *e.g.* in a series of connected links to determine what kind of a line or curve is traced by the motion of the individual parts, in the same way as a cycloidal curve is determined from a rolling circle.

Problem 208.

An oscillating arm or lever AB *is connected by a link* BD *to a crank* DC *revolving round a centre* C, *to determine the path of a curve traced by any point* d' *through a complete double oscillation of the arm* AB, i.e. *from m to h, and back again from h to m.*

With C as centre, and radius CD, draw a complete circle, and divide it into eight equal parts in the points D, E, F, G, H, K, M, and N. Join each of these points to centre C. This will show the crank in eight positions.

With A as centre, and radius AB, draw an arc. With K as centre, and radius equal to BD, set off on this arc the point *k*.

Join K*k*. Repeat this operation from each of the points on the circle in succession; *e.g.* with the same radius BD, set off *m* from M, *n* from N, etc., and join each with its corresponding letter. Join each of the points *k, m, n,* etc., with A. This will show the lever in eight positions. Set off on each of the lines

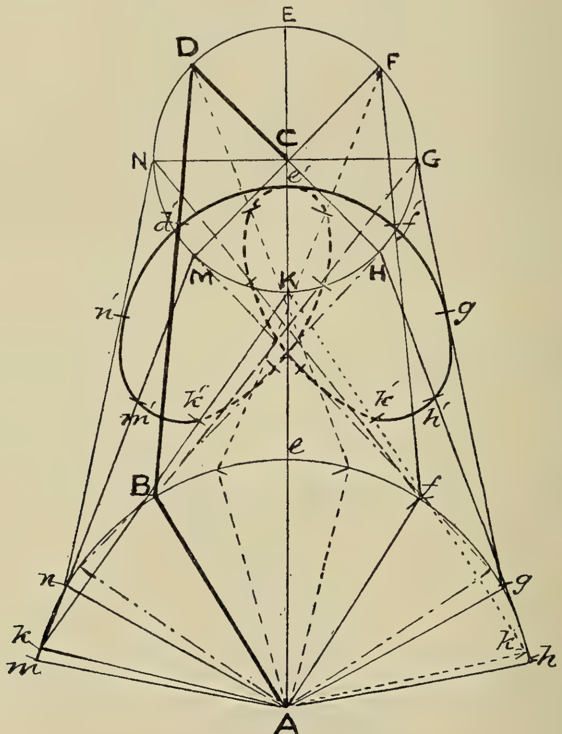


Fig. 322.

K*k*, M*m*, etc., from the points on the circle, a distance equal to Dd', which will give the points e', f', g', h', k', m', and n'. Draw a fair curve through these points. This will complete one-half of the required curve traced by the movement of the lever AB from *m* to *h*. The return from *h* to *m* is found in the same way, and is shown by dotted lines in the diagram. The point k' will trace the dotted part of the curve.

It is a great advantage sometimes to be able to show variations of quantity graphically. This can be done geometrically, by setting off these variations from a standard or fixed base line, according to given or ascertained data.

An irregular line showing the variations of barometric pressure is the most common illustration of this practice.

A cubic foot of air at the earth's surface weighs about 1700 grains; at the height of $3\frac{1}{2}$ miles it would weigh one-half, or 850 grains; and for every additional $3\frac{1}{2}$ miles its weight would be decreased by one-half. From these data construct the following diagram.

Problem 209.

Draw a diagram showing the comparative density of the atmosphere from the surface of the earth to an altitude of 21 *miles.*

Draw a line AB, and divide it into six equal parts. Let A

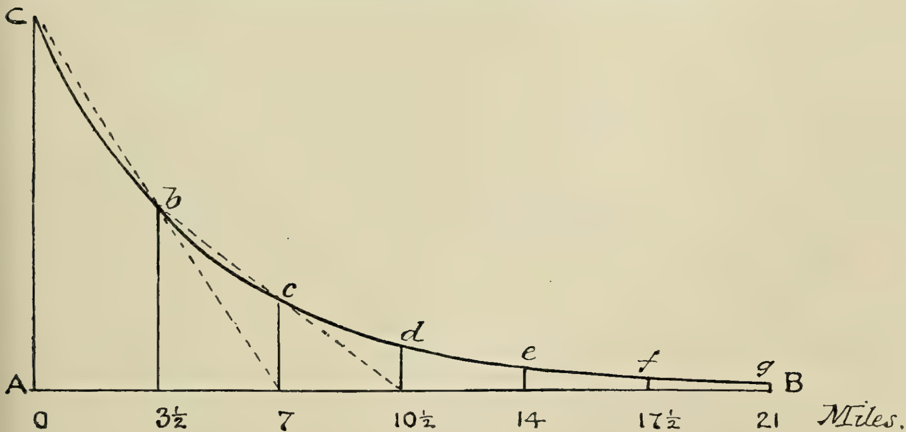


Fig. 323.

be the zero point, and figure each division with a multiple of $3\frac{1}{2}$. These will represent the various altitudes in miles. From each of these points draw the lines C, *b*, *c*, *d*, etc., perpendicular to AB. Let the line AC, which may be drawn to any convenient

length, represent 1700 grains. Make the perpendicular *b* equal to half of AC. This can be done by joining C with the bottom of the perpendicular *c*, as shown by dotted line. Make the perpendicular *c* half the height of perpendicular *b*. In the same way, *d* half of *c*, etc. Draw a fair curve through the top points of these perpendiculars, which will be the curve required.

If we let AC stand for 30 inches, the height of the mercurial column at the earth's surface, then the curve will show the comparative heights of the mercurial column at the different altitudes shown.

EXERCISES IN PLANE GEOMETRY

NOTE.—Feet are represented by one dash ($'$), and inches by two dashes ($''$); 3 feet 6 inches would be written thus—$3'\ 6''$.

CHAPTER I

1. Draw lines of the following lengths: $3''$, $4\frac{1}{2}''$, $2\frac{3}{4}''$, $1\frac{7}{8}''$, $2\cdot25''$, $3\cdot50''$, $1\cdot75''$.

2. Draw an acute angle, and an obtuse angle.

3. Draw the following triangles, viz. equilateral, scalene, isosceles, obtuse-angled, right-angled, and acute-angled.

4. Draw a triangle, and write the following names to its different parts, viz. hypotenuse, vertex, base, median, and altitude.

5. Draw the following figures, viz. rectangle, rhombus, square, trapezoid, rhomboid, trapezium, isosceles trapezoid, and right-angled trapezoid.

6. Draw a circle, and illustrate the different parts, viz. sector, radius, chord, arc, diameter, segment, and tangent.

CHAPTER II

1. Draw two parallel lines $2\frac{3}{4}''$ long and $1\frac{3}{8}''$ apart.

2. Draw a line $3\cdot75''$ long; at the right-hand end erect a perpendicular $2\cdot25''$ high; then, $1\cdot50''$ from it, another perpendicular $1\frac{7}{8}''$ high; and bisect the remaining length by a line $3''$ long, at right angles to it.

3. Draw a line $3\frac{5}{8}''$ long, and divide it into seven equal parts.

4. Draw a line $2\frac{3}{4}''$ long; from the left-hand end mark off a distance equal to $1\frac{1}{4}''$, and from the right-hand end a distance of $\frac{7}{8}''$; draw another line $1\cdot75''$ long, and divide it in the same proportion.

5. Mark the position of three points A, B, and C—A to be $1\frac{3}{4}''$ from B, B to be $2\frac{1}{4}''$ from C, and C $1\frac{7}{8}''$ from A; and join them.

6. Draw an angle equal to the angle ACB in the preceding question, and bisect it.

7. Draw a right angle, and trisect it; on the same figure construct and mark the following angles, viz. 15°, 30°, 45°, 60° ,75°; and also $7\frac{1}{2}°$, $22\frac{1}{2}°$, and $37\frac{1}{2}°$.

8. Construct a triangle with a base $1\frac{3}{4}''$ long; one angle at the base to be 60°, and the side opposite this angle 2″ long.

9. Construct a triangle with a base 2·25″ long, and an altitude of 1·75″.

10. On a base $1\frac{7}{8}''$ long, construct an isosceles triangle; the angle at its vertex to be 30°.

11. Draw a scalene triangle on a base 2″ long; and construct a similar triangle on a base 1·75″ long.

12. On a base $2\frac{1}{8}''$ long draw a triangle with the angle at its vertex 90°.

13. Let a line 2·25″ long represent the diagonal of a rectangle; complete the figure, making its shorter sides $\frac{7}{8}''$ long.

14. Construct a rhombus with sides $1\frac{3}{8}''$ long, and one of its angles 60°.

15. Draw any two converging lines, and through any point between them draw another line which, if produced, would meet in the same point as the other two lines produced.

16. Fix the position of any three points not in the same line, and draw an arc of a circle through them.

17. Draw an arc of a circle, and on it mark the position of any three points; on these points, without using the centre, draw lines which, if produced, would meet in the centre of the circle containing the arc.

18. Construct a triangle, the perimeter to be 5·6″, and its sides in the proportion of 5 : 4 : 3.

Chapter III

1. Inscribed in circles of varying diameters, draw the regular polygons, from a pentagon to a duodecagon, by a general method, and figure the angles formed by their sides.

2. In circles of various radii, draw all the preceding polygons by special methods. Join their angles to the centre of the circle by radii, and figure the angles between the radii.

3. On lines varying in length, draw the same polygons by a general method.

4. Construct on lines of different lengths the same polygons by special methods.

5. Construct an irregular hexagon from the following data: Sides, AB 1″, BC $1\frac{1}{4}$″, CD $\frac{3}{4}$″, DE $1\frac{1}{2}$″, EF $1\frac{1}{4}$″; Angles, ABC 140°, BCD 130°, CDE 110°, DEF 120°.

6. Construct an irregular pentagon from the following data: Sides, AB 1·25″, BC 1·3″, CD 1·2″, DE 1·3″, EA 1·4″; Diagonals, AC 1·8″, AD 1·6″.

7. Construct a regular polygon with one side 1″ in length and one angle 140°.

8. How many degrees are there in each of the angles at the centre of a nonagon?

9. Construct a regular polygon on the chord of an arc of 72°.

10. Inscribe in any given circle an irregular heptagon whose angles at the centre are respectively 52°, 73°, 45°, 63°, 22°, 36°, and 69°.

Chapter IV

1. Within a given circle inscribe a square, and about the same circle describe an equilateral triangle.

2. Construct a rhombus with sides $1\frac{1}{2}$″ long, and its shorter diagonal 1·75″; inscribe a circle within it, and let the circle circumscribe an equilateral triangle.

3. Construct a trapezium with two of its sides $1\frac{3}{4}$″ and two $1\frac{1}{4}$″ respectively, and with its longer diagonal $2\frac{1}{2}$″; inscribe within it a square, and let the square circumscribe an equilateral triangle.

4. Draw a regular hexagon with 1″ sides, and let it circumscribe a square; inscribe a regular octagon within the square.

5. Draw any triangle, and describe a circle about it.

6. Construct a square of $2\frac{1}{2}$″ sides, and in it inscribe an isosceles triangle with a $1\frac{3}{4}$″ base; inscribe within the triangle a rectangle, one side of which is $1\frac{1}{4}$″.

7. Within a square of 1·75″ sides, inscribe an isosceles triangle with angle at vertex 60°; inscribe a circle within the triangle.

8. Within a circle of any radius, inscribe a regular duodecagon, and let it circumscribe a hexagon.

9. Construct a rhomboid with sides 2″ and $1\frac{1}{2}$″, its contained angle to be 60°; inscribe within it a rhombus.

10. Within an equilateral triangle of 3″ sides, inscribe a circle, and within it 3 equal circles.

11. Draw a circle of $1\frac{1}{2}$″ radius; inscribe within it an equilateral triangle; inscribe within the triangle three equal circles touching each other and each one side of the triangle only.

12. Construct a square of 2″ sides, and let it circumscribe four equal circles; each circle to touch two others, as well as two sides of the square.

13. Construct a square with sides of 2·3″, and inscribe four equal circles within it; each circle to touch two others, as well as one side only of the square.

14. Within a triangle of 2·7″ sides, inscribe six equal circles.

15. Within a circle of 1·7″ radius, inscribe seven equal circles.

16. Draw two concentric circles, and between them, six equal circles, to touch each other as well as the two concentric circles.

17. In a decagon, inscribe five equal circles.

Chapter V

1. Construct an equilateral triangle of $1\frac{1}{2}$″ sides, and about it describe a trefoil having tangential arcs.

2. Construct a pentagon of $\frac{3}{4}$″ sides, and about it describe a cinquefoil having adjacent diameters.

3. Draw a pentagon of $\frac{1}{2}$″ sides, and about it construct a cinquefoil having tangential arcs.

4. In a circle of $1\frac{1}{2}$″ diameter, draw nine equal semicircles having adjacent diameters.

5. In a circle of 1″ diameter, inscribe a quatrefoil having tangential arcs.

6. Construct a regular decagon in a circle of $\frac{3}{4}$″ radius, and within it inscribe a cinquefoil.

Note.—Foiled figures can be inscribed in all the regular polygons that have an even number of sides, by first dividing them into trapezia and then proceeding by the method shown in Prob. 65; or they can be drawn in circles divided into any number of equal sectors, by Prob. 106.

Chapter VI

1. Draw a circle 1·7″ in diameter; at any point in its circumference, draw a tangent.

2. Draw a circle 1·25″ in radius; from a point one inch outside the circle, draw a tangent to it.

3. Draw a circle $1\frac{7}{8}''$ in diameter; from any point outside the circle, draw two tangents without using the centre.

4. Draw two lines, enclosing an angle of 45°; draw a circle $1\frac{1}{2}''$ in diameter, tangential to these lines.

5. At any point in the arc of a circle draw a tangent, without using the centre.

6. Draw two circles of 2″ and 1″ radii, with their centres 3″ apart; draw transverse common tangents to them.

7. Draw two circles of 1·7″ and 1″ radii, with their centres 2·75″ apart; draw direct common tangents to them.

8. Draw two lines at an angle of 30° with each other, and a third line cutting them both at any convenient angle; draw two circles tangential to all the three lines.

9. Construct a triangle with sides 2·25″, 1·8″, and 1·25″; draw an inscribed and three escribed circles tangential to the lines forming the sides.

10. Draw two lines enclosing an angle of 45°; fix a point in any convenient position between these two lines, and draw a circle that shall pass through this point and be tangential to the two lines.

11. Draw a circle 1·25″ in diameter, and half an inch from it draw a straight line; draw a circle of $\frac{3}{8}''$ radius, tangential to both the circle and the line.

12. Draw two circles of 1″ and $\frac{1}{2}''$ radius, with their centres $2\frac{1}{2}''$ apart; draw another circle tangential to both externally.

13. Draw two circles 1·50″ and 1″ in diameter, their centres to be

2·25″ apart; draw another circle $3\frac{1}{2}''$ in diameter, touching the larger circle externally, and the smaller one internally.

Chapter VII

1. Draw three lines 1·25″, 2·3″, and 2·75″ respectively, and find their greater fourth proportional.

2. Draw two lines 2·7″ and 1·5″ in length, and find their less third proportional.

3. Draw a line 2·5″ in length, and produce it so that its extra length shall be in proportion to its original length as 3 : 5.

4. Draw two lines $2\frac{1}{4}''$ and $\frac{7}{8}''$ in length, and find their mean proportional.

5. Construct a triangle on a base 3·25″ in length, so that its three angles are in the proportion of 3, 4, and 5.

6. Divide a line 3·4″ in length into extreme and mean proportion.

7. Divide a line 2·75″ in length so that one part is in proportion to the other as 2 : 4.

8. Draw two lines 1·25″ and 2·3″ respectively, and find their greater third proportional.

9. Draw three lines 3″, $2\frac{1}{4}''$, and $1\frac{7}{8}''$ in length, and find their less fourth proportional.

Chapter VIII

1. Draw a right-angled triangle, with a base of $1\frac{1}{4}''$ and an altitude of 2″; on the same base construct an isosceles triangle of equal area.

2. Draw a square of 1·75″ sides; and construct a rhomboid of equal area, two of its angles to be 60°.

3. Draw a regular heptagon on a $\frac{3}{4}''$ side; and construct an isosceles triangle equal to it in area.

4. Construct a triangle with sides of 2·75″, 1·75″, and 2″; from any point within it, trisect its area.

5. Construct a square having an area of five square inches, and make an isosceles triangle equal to it.

6. Draw a pentagon on a 1·5″ side; and construct a rectangle equal to it on a 3″ side.

7. Construct a trapezium with sides of $1\frac{1}{4}''$, $2\frac{5}{8}''$, $3\frac{1}{4}''$, and $2\frac{1}{2}''$—one of its angles to be 60°. Through one of its angles bisect it.

8. Construct any irregular octagon, and divide it into seven equal parts.

Chapter IX

1. Construct a square equal in area to the sum of two other squares, of $1''$ and $1\frac{1}{2}''$ sides respectively.

2. Construct an equilateral triangle equal in area to the difference between two other equilateral triangles with sides of $1{\cdot}5''$ and $2{\cdot}7''$ respectively.

3. Draw a circle equal in area to the sum of two other circles of $\frac{3}{4}''$ and $1\frac{1}{4}''$ diameters.

4. Draw a regular heptagon on a side of $1{\cdot}5''$; and construct a similar polygon three-fifths of its area.

5. Construct an equilateral triangle on a side of $2\frac{1}{2}''$; draw a similar triangle four-fifths of its area, and divide it into two equal parts by a line parallel to one of its sides.

6. Draw a circle three-fourths the area of a given circle, and divide it by concentric circles into three equal parts.

Chapter X

1. Construct a scale to measure feet and inches; the R.F. to be $\frac{1}{36}$, and its scale length value 15 feet.

2. Construct a scale to measure yards and feet, the R.F. to be $\frac{1}{72}$, to measure 18 yards.

3. Construct a diagonal scale to measure feet and inches, R.F. $\frac{1}{72}$, to measure 36 feet. Take off a length of 17′ 9″.

4. On a map, a distance known to be 20 miles measures $10''$; construct a diagonal scale to measure miles and furlongs, long enough to measure 12 miles.

5. Construct a diagonal scale to measure yards and feet, R.F. $\frac{1}{180}$, to measure 30 yards.

6. On a map showing a scale of kilomètres, 60 are found to equal $3''$. What is the R.F.? Construct a comparative scale of English miles, to measure 100 miles.

CHAPTER XI

1. On a line 4″ long, draw a semicircle, and upon it set out the primary divisions of a protractor, by construction alone.

2. Construct a scale of chords from the protractor set out in the preceding question.

CHAPTER XIV

1. Draw a regular hexagon on a side of 1″; construct a similar figure on a side of $\frac{1}{2}$″, using one angle as the centre of similitude.

2. Draw a rectangle with sides of 2·7″ and 1·5″; construct a similar figure by inverse similitude, with sides in the proportion to those given as 3 : 5.

3. Draw a regular pentagon in a circle of $2\frac{3}{8}$″ diameter; construct a similar figure by direct similitude, with sides in the proportion to those of first pentagon as 4 : 7.

CHAPTER XV

1. Construct an ellipse: major axis 3·75″, minor axis 2·25″. Select any point in the curve, and draw a tangent to it.

2. Construct an ellipse; the foci to be $2\frac{1}{4}$″ apart, and the transverse diameter $3\frac{3}{4}$″.

3. Draw a rectangle 3·25″ × 2·3″, and inscribe an ellipse within it.

4. Draw a parallelogram $3\frac{7}{8}$″ × $2\frac{3}{8}$″, two of its angles to be 60°; inscribe an ellipse within it.

5. Construct an ellipse by means of a paper trammel; the transverse diameter being $4\frac{1}{2}$″, and the conjugate diameter 3″ (Prob. 178).

6. Draw the two diameters of an ellipse each 3″ long, and at an angle of 45° with each other; complete the ellipse.

7. Make a tracing of the ellipse given in question 3, and find the diameters, foci, tangent, and normal.

8. With an abscissa 3″ long and an ordinate 2″ long, construct a parabola.

9. With a diameter 1·4″, an ordinate 1·8″, and an abscissa 1·4″, construct an hyperbola.

10. Draw a rectangle $3'' \times 2''$, and let two adjacent sides represent the axes of a rectangular hyperbola ; measure off along one of its longer edges $\frac{1}{2}''$, and let this point represent the vertex of the curve ; complete the hyperbola.

11. Draw a line $4''$ long to represent an abscissa of a parabola ; at one end draw a line $3''$ long, at right angles to it, to represent the directrix ; from the directrix, along the abscissa, set off $1''$ to mark the focus ; complete the parabola. At any point in the curve, draw a tangent and normal to it.

Chapter XVI

1. In a circle of $2''$ radius, construct an Archimedean spiral of 3 revolutions.

2. With a rolling circle $2''$ in diameter and a generating point $1\frac{1}{2}''$ from its centre, construct a trochoid. At any point in the curve, draw a tangent and normal.

3. With a generator $1{\cdot}6''$ in diameter, and a director of $3{\cdot}4''$ radius, construct an epicycloid. At any point in the curve, draw a tangent and normal.

4. Starting with a radius vector $2''$ long, find points for two complete convolutions of a logarithmic spiral ; the angle between each successive position of the radius vector to be $20°$. Draw a fair curve through the points found. At any point in the curve, draw a tangent and normal.

5. Draw the involute of a circle $2{\cdot}75''$ in diameter. At any point in the curve, draw a tangent and normal.

6. Construct a cissoid to a circle $2{\cdot}6''$ in diameter.

7. Construct an inferior conchoid ; the pole to be $1{\cdot}7''$ above the asymptote, and the diameter $\frac{3}{4}''$.

8. With a generator $1{\cdot}8''$ in diameter and a director of $4{\cdot}7''$ radius, draw the hypocycloid. At any point in the curve, draw a tangent and normal.

SOLID GEOMETRY

CHAPTER XVII

INTRODUCTION

In the preceding subject, Plane Geometry, we have been restricted to figures having length and breadth only, but Solid Geometry treats of figures that have thickness in addition to length and breadth.

Practical Solid Geometry is a branch of *Orthographic Projection*, and is frequently taken up after that subject; but as it is somewhat easier for a student to comprehend, especially in the elementary stage, it will be better, perhaps, to take it first. It is easier for this reason: the solid figure assists the eye and enables the student to understand better the direction and relation between lines and planes than would be possible without this help.

The objects taken to illustrate the principle of this subject are described under the head of *Definitions* (Chap. I.)

By means of Practical Solid Geometry we are enabled to represent on a plane—such as a sheet of paper—solid objects in various positions, with their relative proportions, to a given scale.

Let us take some familiar object, a dressing-case for instance, ABCD, Fig. 1, and having procured a stiff piece of drawing-paper HKLM, fold it in a line at X, parallel to one of its edges; then open it at a right angle, so that HX will represent the edge of a *vertical plane*, and XL the edge of a *horizontal plane;*

the line at X, where the two planes intersect, is called the *line of intersection, intersecting line*, or *ground line*; it shows where the two planes intersect each other, and is generally expressed by the letters X and Y, one at each end.

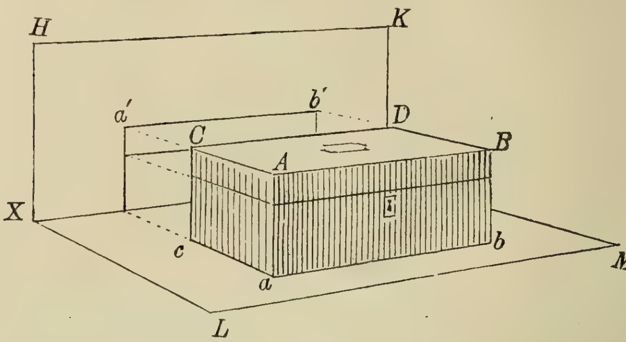


Fig. 1.

Having placed the dressing-case on the horizontal plane, with its back parallel to the vertical plane, let us take a pencil

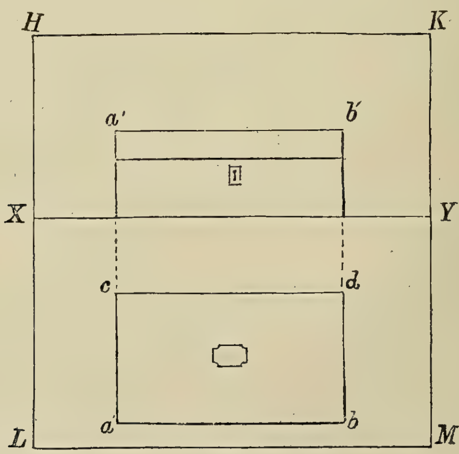


Fig. 2.

and trace its position on the *horizontal plane* by drawing a line along its lower edges; also its shape on the *vertical plane*. This can be done by placing the eye directly opposite each of its

front corners in succession and marking their apparent position on the vertical plane, and joining them. Having done this, we will remove the dressing-case and spread the paper out flat upon a table: this is shown in Fig. 2. We have now two distinct views of the object. The lower one is called a PLAN, and represents the space covered by the object on the horizontal plane, or a view of the dressing-case seen from above. The upper view shows the space covered on the vertical plane, and is called an ELEVATION: it represents the front view of the object.

In Solid Geometry all objects are represented as they would

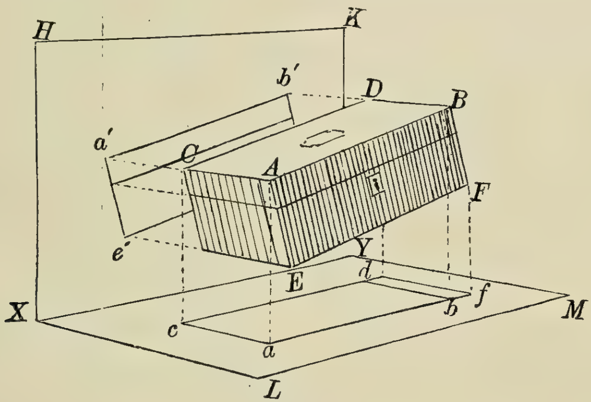


Fig. 3.

appear traced or projected on these two planes at right angles to each other: they are called *co-ordinate planes*. It is not necessary that the object should be parallel to them, as in Fig. 1: we can arrange it in any position, making any possible angle with either plane, but the line connecting the point on the object with its respective plane must always be perpendicular to that plane. We shall understand this better if we refer to Fig. 3, in which we will imagine the dressing-case suspended in mid-air, with its back still parallel to the vertical plane, but its under side inclined to the horizontal plane. We will now trace it on each plane as before, then by spreading the paper out flat we get a drawing as shown in Fig. 4.

The student should compare Fig. 1 with Fig. 2, as well as Fig. 3 with Fig. 4, so as to thoroughly understand the relation between the co-ordinate planes.

The lines Aa', Bb', Ee' (Fig. 3), are all perpendicular to the vertical plane; and the lines Aa, Bb, Cc, etc., are perpendicular to the horizontal plane. These lines are called *projectors,* and

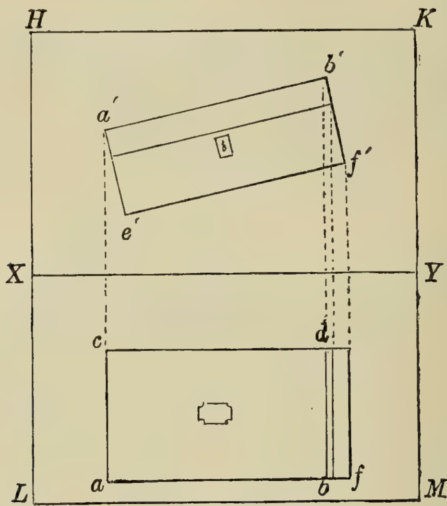


Fig. 4.

are here represented by dotted lines. The points in which these lines meet the two co-ordinate planes are called *projections:* if they are on the vertical plane they are called *vertical projections,* and if on the horizontal plane *horizontal projections,* of the different points; *e.g.* a' is the *vertical projection* of point A, and a is its *horizontal projection.* The length of the horizontal projector shows the distance of the point from the vertical plane, and the length of the vertical projector its distance from the horizontal plane.

All through this subject the points of the object are distinguished by capital letters, as A, B, etc., while their horizontal projections are represented as a, b, etc., and their vertical pro-

jections as a', b', etc.; by this means we are enabled to distinguish the plan from the elevation. V.P. will also be used to express the *vertical plane,* and H.P. the *horizontal plane;* the letters XY will always stand for the *ground line.*

The student should take particular notice that the lower points in the plan always represent the front points in the elevation.

It is not necessary to have an object to trace; if we know its dimensions, and its distances from the two planes, we can construct the drawings as shown in Figs. 2 and 4.

It is impossible to see an object as it is represented by orthographic projection, because the eye is supposed to be directly opposite each point at the same time. As the positions of the points are found by projectors parallel to each other, it is called *parallel projection.*

Figs. 1 and 3 are perspective views, and Figs. 2 and 4 are geometrical drawings of the same object. If the latter were drawn to scale, we could find out the length, breadth, and thickness of the object from these drawings.

Each perspective view is supposed to be taken from one fixed point, *i.e.* the eye; and lines drawn from different parts of the object converge towards the eye considered as a point. These lines represent rays of light from the object, and are called visual rays: they form a cone, the vertex of which is the position of the eye; consequently, Perspective is called *Conical, Radial,* or *Natural Projection,* because it represents objects as they appear in nature.

There are other kinds of projection, viz. *Stereographic, Orthogonal, Isometric,* and *Horizontal,* but these will be described later.

We will now take four simple solids, viz. a *cube,* a *rectangular solid,* a *pyramid,* and a *triangular prism,* and show the different positions they can occupy with reference to the co-ordinate planes, *i.e.* the V.P. and H.P.

Fig. 5 represents the four solids in what is called *simple positions,* i.e. *parallel to both the* V.P. *and* H.P.

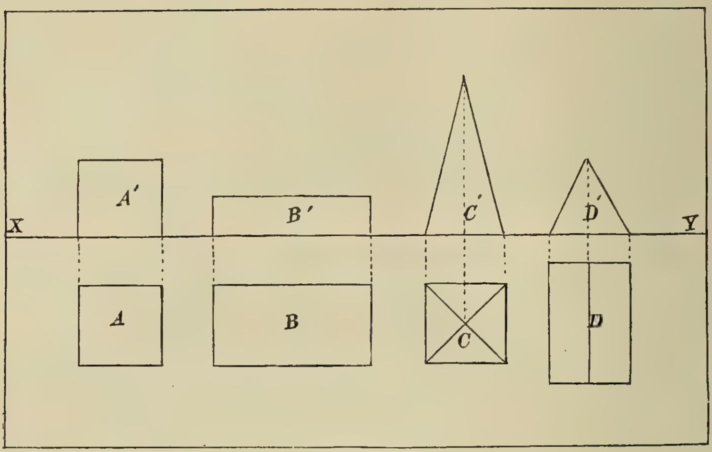


Fig. 5.

A is the plan of the		*cube*	and	A′	its elevation.
B	„	*rectangular solid*	„	B′	„
C	„	*pyramid*	„	C′	„
D	„	*triangular prism*	„	D′	„

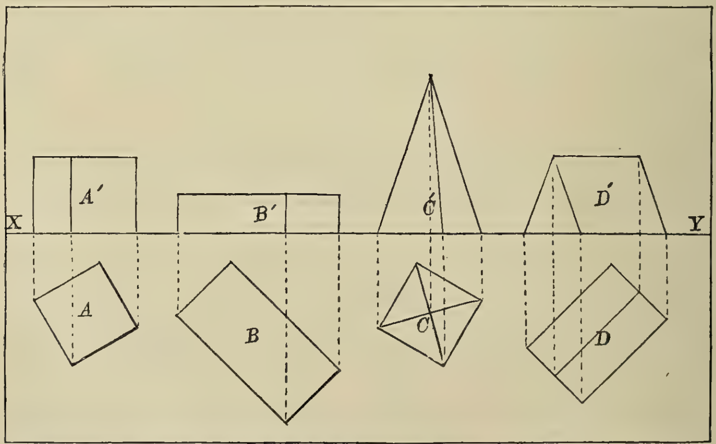


Fig. 6.

Fig. 6 represents the same solids with their bases *on the* H.P. as before, but their sides are now *inclined to the* V.P.

Fig. 7 shows them with their fronts and backs *parallel to the* V.P., as in Fig. 5, but with their bases *inclined to the* H.P.

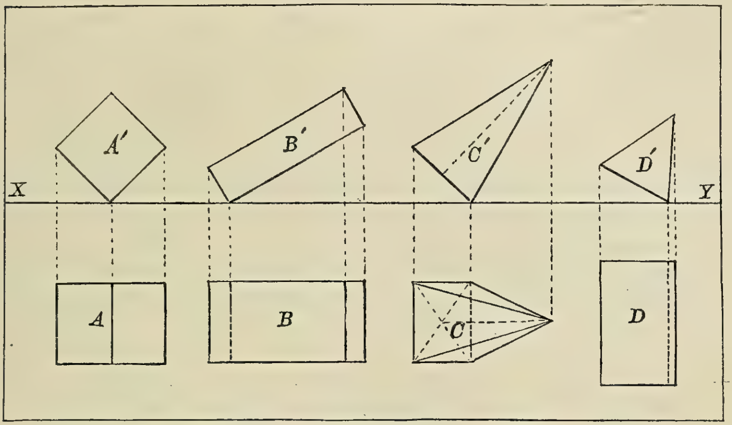


Fig. 7.

Fig. 8.—They are here represented *inclined to both the* V.P. *and* H.P., but they still have one set of edges parallel to the H.P.

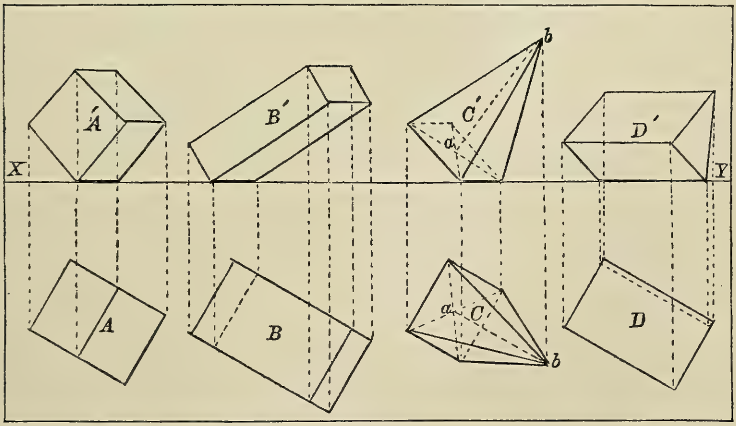


Fig. 8.

Fig. 9.—Here they are shown with every line inclined to both planes: instead of having one edge resting on the H.P., as in Fig. 8, they are each poised on a corner. To distinguish this

position from the one illustrated in Fig. 8, we will call it *com-*

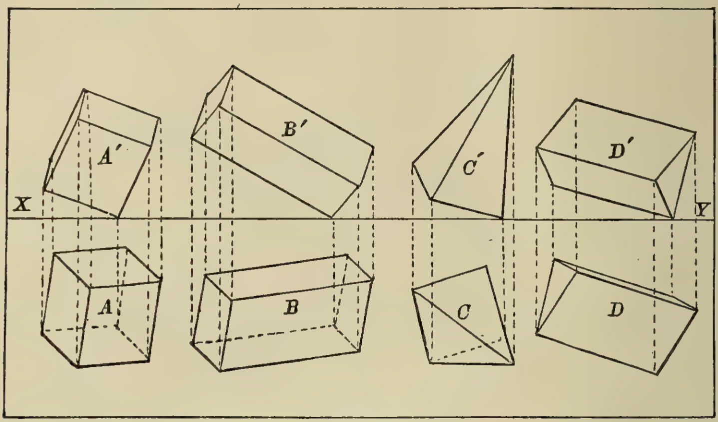


Fig. 9.

pound oblique; although Figs. 8 and 9 generally come under one head, as objects inclined to both planes.

CHAPTER XVIII

SIMPLE SOLIDS IN GIVEN POSITIONS TO SCALE

Note.—Feet are indicated by one dash, and inches by two dashes: thus—3′ 6″ represents 3 feet 6 inches.

Problem 1.

To project a quadrilateral prism 5″ × $2\frac{1}{2}$″ × $2\frac{1}{2}$″ *with one of its smaller faces on the* H.P., *parallel to the* V.P., *and* $\frac{3}{4}$″ *from it. Scale* $\frac{1}{4}$ *full size.* Fig. 10.

First draw the line XY; then draw the plan *abcd* $\frac{3}{4}$ inch

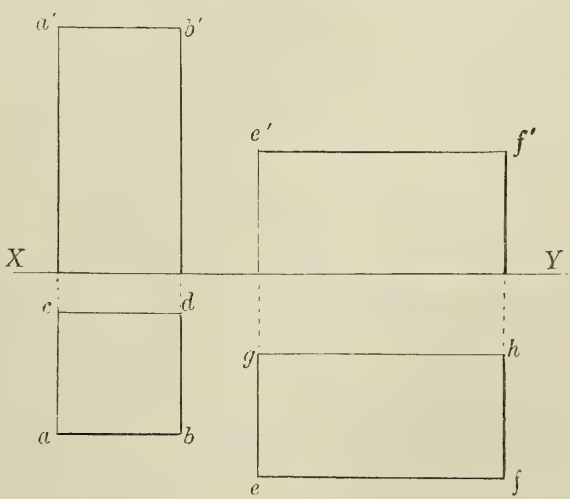


Fig. 10. Fig. 11.

below it. Draw perpendicular lines above XY, immediately

over the points a and b, 5″ in height, which give the points a' and b'. Join $a'b'$. This is the elevation of the solid.

Problem 2.

To project the same solid with one of its longer faces resting on the H.P., *parallel to the* V.P., *and* $1\frac{3}{4}''$ *from it, to the same scale.* Fig. 11.

Draw the plan $efgh$, $5'' \times 2\frac{1}{2}''$, and $1\frac{3}{4}$ inches below XY. Draw perpendiculars above XY, $2\frac{1}{2}''$ high, and directly over the points e and f, which will give the points e' and f'. Join e' and f', which completes the elevation.

The student should now project the four solids illustrated in Fig. 5 in the positions there shown, but to the following dimensions and scale :—

A	to have	a base	$4'' \times 4''$,	to be	$4''$	high and	$2\frac{1}{2}''$	from	V.P.
B	,,	,,	$8'' \times 4''$	,,	$2''$	,,	$2\frac{1}{2}''$	,,	V.P.
C	,,	,,	$4'' \times 4''$	,,	$8''$	,,	$2\frac{1}{2}''$	,,	V.P.
D	,,	,,	$6'' \times 4''$	,,	$4''$	,,	$1\frac{1}{2}''$	,,	V.P.

Scale $\frac{1}{4}$ full size.

Problem 3.

To project a quadrilateral prism $10'' \times 5'' \times 5''$, *with one of its smaller faces on the* H.P., *at an angle of* 45° *with* V.P., *and one edge* $3\frac{1}{4}''$ *from* V.P. *Scale* $\frac{1}{8}$ *full size.* Fig. 12.

Draw XY. Take the point d, $3\frac{1}{4}''$ below XY, and draw the square $abcd$ below this point with its sides at an angle of 45° with XY. This will be the plan. Erect perpendiculars 10″ high above XY, and directly over the points a, b, and c. Join the tops of these perpendiculars, which completes the elevation.

PROBLEM 4.

To project the same solid lying on one of its longer faces on the H.P. *with its longer edges forming an angle of* 30° *with the* V.P., *and one of its corners* $1\frac{3}{4}''$ *from* V.P., *to the same scale.* Fig. 13.

Draw the point *h*, $1\frac{3}{4}''$ below XY, and construct the plan *efgh* at the required angle below this point. Erect perpendiculars 5″

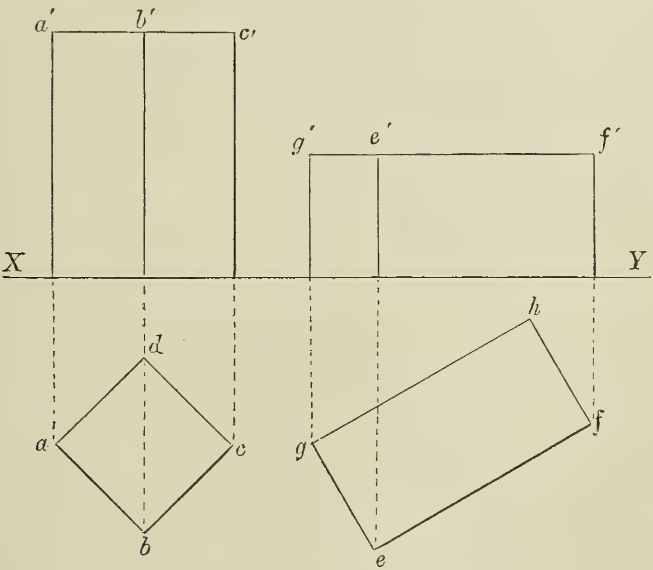


Fig. 12. Fig. 13.

high above XY, immediately over the points *g*, *e*, and *f*. Join the tops of these perpendiculars, which completes the elevation.

The student should now project the four solids illustrated in Fig. 6, in the positions there shown, but to the following dimensions and scale :—

A to have a base 5″ × 5″, to be 5″ high, with one side inclined at an angle of 30° with the V.P., and 2″ from it.

B to have a base 10″ × 5″, to be $2\frac{1}{2}''$ high, with both sides inclined at an angle of 45° with the V.P., and 2″ from it.

C to have a base 5″ × 5″, to be 10″ high, with one edge of the base

making an angle of 60° with the V.P., and its nearest point 2″ from it.

D to have a base 8″ × 5″, to be 5″ high, with both sides of its base making an angle of 45° with the V.P., and the nearest point 2″ from it.

Scale $\frac{1}{3}$ full size.

Problem 5.

To project a quadrilateral prism $7\frac{1}{2}'' \times 3\frac{3}{4}'' \times 3\frac{3}{4}''$, *resting on one of its shorter edges on the* H.P., *and with its longer edges parallel to the* V.P., *but inclined at an angle of* 60° *to the* H.P.; *one of its faces to be* $2\frac{1}{2}''$ *from* V.P. *Scale* $\frac{1}{6}$ *full size.* Fig. 14.

Draw XY. At point e' draw the elevation $e'a'b'c'$ at the

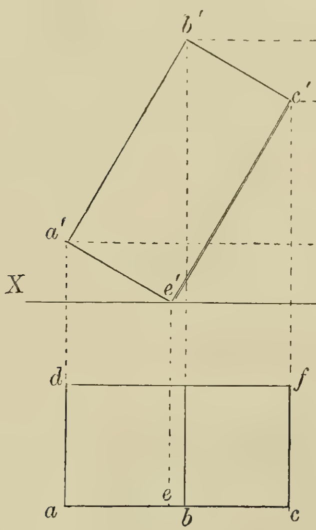


Fig. 14.

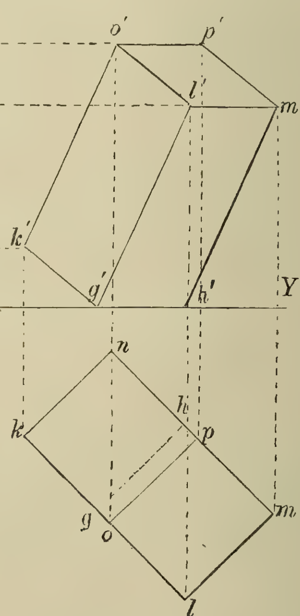


Fig. 15.

required angle. $2\frac{1}{2}''$ below XY draw the line df parallel to it. Let fall lines from a', b', and c', at right angles to XY, and make da and fc each $3\frac{3}{4}''$ long. Join ac, which completes the plan.

The student should now project the four solids illustrated in Fig. 7 from the following conditions :—

A to be $2\frac{1}{2}'' \times 2\frac{1}{2}'' \times 2\frac{1}{2}''$, with its base inclined at an angle of 45° to H.P.; to be parallel to the V.P., and $2\frac{1}{2}''$ from it.

B to be $5'' \times 2\frac{1}{2}'' \times 1\frac{1}{4}''$, with its base inclined at an angle of 30° to H.P.; to be parallel to the V.P., and $2\frac{1}{2}''$ from it.

C to be $2\frac{1}{2}'' \times 2\frac{1}{2}'' \times 5''$, with its base inclined at an angle of 45° to H.P.; to have the edge of its base parallel to the V.P., and $2\frac{1}{2}''$ from it.

D to be $4'' \times 2\frac{1}{2}'' \times 2\frac{1}{2}''$, with its base inclined at an angle of 30° to H.P.; to have its ends parallel to V.P., and $1\frac{3}{4}''$ from it.

Scale $\frac{1}{2}$ full size.

Before proceeding with the next problem, Fig. 15, it will be necessary to understand thoroughly the angles which the solid forms with the co-ordinate planes.

The longer edges are still inclined to the H.P. at an angle of 60°; but instead of being parallel to the V.P., as in Fig. 14, *they are in planes* inclined to the V.P. at an angle of 45°. This does not mean that they form an angle of 45° with the V.P. Let us illustrate this with a model.

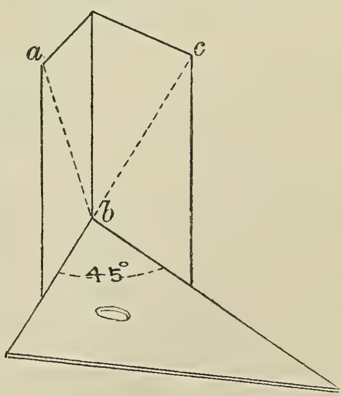


Fig. 16.

Take a sheet of notepaper and draw a diagonal to each of its inside pages, as *ab* and *bc*, Fig. 16. Now stand it on a table against the 45° angle of a set-square. The two pages will then represent two planes at an angle of 45° with each other. Let the page *a* represent the V.P., and the line *bc* one of the edges of the solid. The angle which *bc* makes with the page *a* will be considerably less than 45°.

Problem 6.

To project a quadrilateral prism $7\frac{1}{2}'' \times 3\frac{3}{4}'' \times 3\frac{3}{4}''$, *resting on one of its shorter edges on the* H.P., *with its longer edges inclined to the* H.P. *at an angle of* 60°, *and in vertical planes inclined to the* V.P. *at an angle of* 45°; *one of its lower corners to be* $1\frac{1}{4}''$ *from* V.P. *Scale* $\frac{1}{6}$ *full size.* Fig. 15.

Find the position of point n, $1\frac{1}{4}''$ below XY, and draw the lines nm and nk at an angle of 45° to XY. Make nm and nk equal in length to df and da (Fig. 14). Draw the line kl parallel to nm, and of the same length. Join lm. Make kg and lo equal to ac and cb (Fig. 14). Draw gh and op perpendicular to kl. This will complete the plan. As every point in the elevation is found in precisely the same way, it is only necessary to explain the projection of one point: o', for example.

Draw a perpendicular from o on plan till it meets a horizontal line drawn from b' (Fig. 14). This gives the position of point o'.

The student should now project the four solids shown in Fig. 8 from the following conditions:—

A to be $4'' \times 4'' \times 4''$, with one set of edges parallel to H.P.; its other edges to be inclined to H.P. at an angle of 45°, and in vertical planes inclined to V.P. at an angle of 30°; its nearest corner to be 4″ from V.P.

B to be $8'' \times 4'' \times 2''$, with one set of edges parallel to H.P.; its longest edges inclined at an angle of 30° to H.P., but which, with its shortest edges, are to be in vertical planes inclined to the V.P. at an angle of 30°. Its nearest corner to be 4″ from V.P.

C to be $4'' \times 4'' \times 8''$; its base to be inclined to the H.P. at an angle of 45°; its axis in a vertical plane inclined to V.P. at an angle of 30°; and its nearest corner 4″ from V.P.

Note.—The axis is a line drawn from the vertex to the centre of the base; as ab, Fig. 8.

D to be $6'' \times 4'' \times 4''$, to have one side inclined at an angle of 30°

with H.P., and its ends in a vertical plane inclined to V.P. at an angle of 60°. The nearest corner to be 4″ from V.P. Scale $\frac{1}{4}$ full size.

Note.—The heights in these elevations are obtained by first drawing side views of the objects, as shown in Fig. 7. The connection is fully shown in Figs. 14 and 15.

Problem 7.

To project a quadrilateral prism $6\frac{1}{4}'' \times 3\frac{1}{2}'' \times 3\frac{1}{2}''$ *resting on one corner on* H.P., *and its faces forming equal angles with it, with its longer edges inclined at an angle of* 60° *to* H.P., *and parallel to* V.P. *Its nearest edge to be* $1\frac{1}{4}''$ *from* V.P. *Scale* $\frac{1}{5}$ *full size.* Fig. 17.

Draw XY, and $1\frac{1}{4}''$ below it draw the line *mn*. In any con-

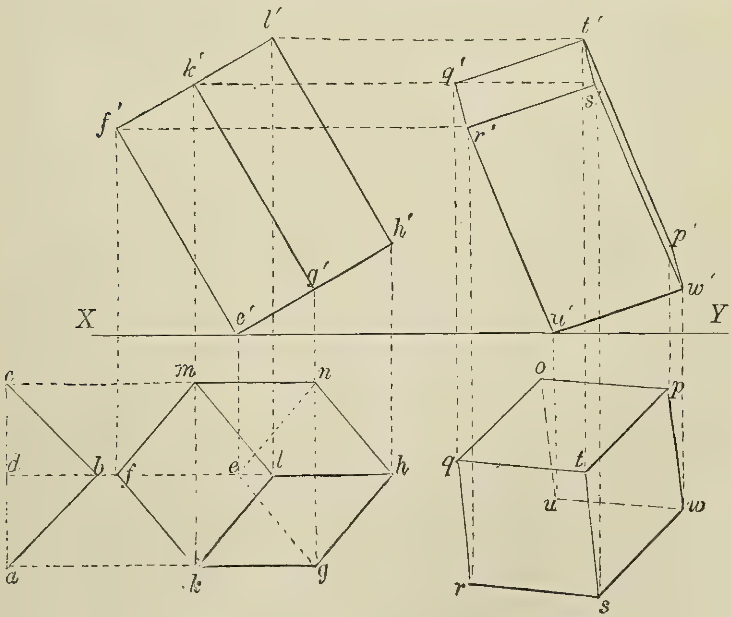


Fig. 17. Fig. 18.

venient position draw the line *ca* perpendicular to *mn*, and from *c* draw *cb* $3\frac{1}{2}''$ long, at an angle of 45°. From *b* as centre,

and with radius *bc*, draw an arc cutting *ca* in *a*. Join *ba*. Also draw *bd* perpendicular to *ca*. This represents one-half of the actual shape of the base of the prism.

At any point c' on XY draw the line $c'f'$ $6\frac{1}{4}''$ long at an angle of 60° to XY, and the line $c'h'$ perpendicular to it. From c', along $c'h'$, set off the distances e', g', h', equal to those of *a*, *d*, *c*. At each of these points draw lines parallel to $c'f'$, and equal to it. Join the tops of these lines. This completes the elevation. Draw lines from *b* and *a* parallel to *mn*. Every point in the plan must come on one of these three lines. Drop a line from f' at right angles to XY till it meets the horizontal line from *b*; this gives the point *f*. Every other point in the plan is found in the same manner.

Problem 8.

To project the same prism, with its longer edges still inclined at an angle of 60° *with* H.P., *and its faces making equal angles with it; but instead of being parallel to* V.P., *as in Fig.* 17, *let them be in vertical planes, inclined at an angle of* 45° *with* V.P. *The nearest corner of prism to be* $1\frac{1}{4}''$ *from* V.P. *Scale* $\frac{1}{5}$ *full size.* Fig. 18.

Note.—We can always imagine any line to be contained in a vertical plane, whether the object contains one or not. In Fig. 15 the line $k'o'$ is contained in the vertical plane $k'o'l'g'$; but in this instance $s'w'$ (Fig. 18) is not contained in one, as the solid does not contain a vertical plane.

At any point *o* $1\frac{1}{4}''$ below XY, draw *oq* at an angle of 45° with XY. Make *oq* equal in length to *mn* (Fig. 17). The plan Fig. 18 is precisely similar in shape to the plan Fig. 17, but turned to make an angle of 45° with XY; so if we take the line *qo* to represent the line *mn*, we can complete the plan from Fig. 17.

Every point in the elevation is found in the same way: erect a perpendicular upon point *r* till it meets a horizontal line drawn from f' (Fig. 17); this gives the point r', and so on till the elevation is completed.

Problem 9.

To project a regular hexagonal prism 10″ *long, and with sides* $3\frac{1}{4}''$ *wide, standing on its base on* H.P., *with one face parallel to* V.P. *and* $2\frac{1}{2}''$ *from it. Scale* $\frac{1}{8}$ *full size.* Fig. 19.

Draw the line XY, and $2\frac{1}{2}''$ below it draw the line *ab*. Complete the hexagon. Above XY draw perpendiculars 10″ long immediately over the points *c, e, f, d*. Join the tops of these perpendiculars, which completes the elevation.

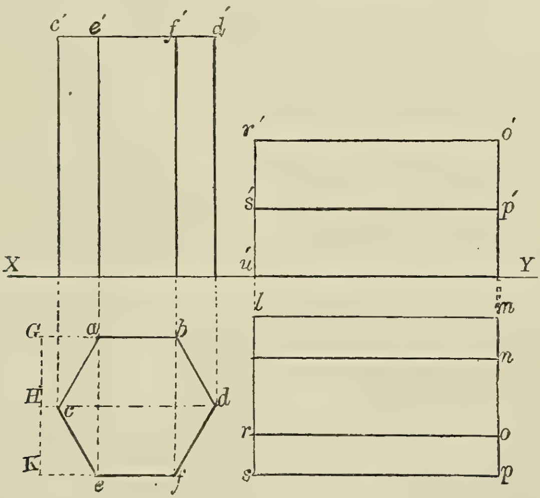


Fig. 19. Fig. 20.

Problem 10.

To project the same prism lying on one face on the H.P., *with its longer edges parallel to the* V.P.; *its nearest edge to be* $1\frac{3}{4}''$ *from* V.P. *Scale* $\frac{1}{8}$ *full size.* Fig. 20.

Draw the line *lm*, 10″ long, $1\frac{3}{4}''$ below XY and parallel to it. Draw the lines *ls* and *mp* perpendicular to *lm*. Set off the distances *m, n, o, p* on *mp* equal to the distances *c′, e′, f′, d′* (Fig. 19). Draw lines from *n, o,* and *p*, parallel to *lm*, till they meet the line *ls*. This completes the plan.

Draw perpendiculars above XY immediately above the points s, p, and set off the distances u', s', r' equal to the distances K, H, G (Fig. 19). Draw lines from the points s' and r' parallel to XY till they meet the perpendicular $p'o'$. This completes the elevation.

Problem 11.

To project a regular hexagonal prism $7\frac{1}{2}''$ *long, and with sides* $2\frac{1}{2}''$ *wide, standing on its base on the* H.P., *with one face inclined to the* V.P. *at an angle of* 45°. *Its nearest edge to be* $1\frac{3}{4}''$ *from* V.P. *Scale* $\frac{1}{6}$ *full size.* Fig. 21.

Draw the line XY, and $1\frac{3}{4}''$ below it mark the position of point

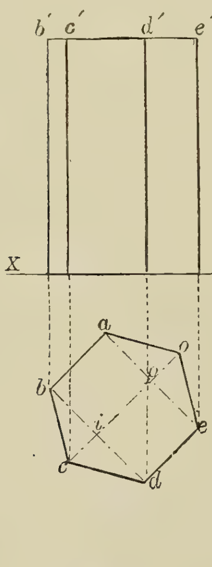


Fig. 21.

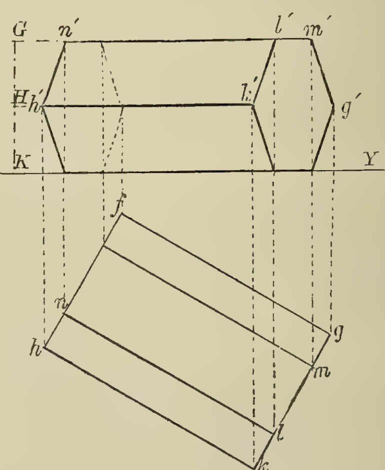


Fig. 22.

a. From a draw the line ab, $1\frac{3}{4}''$ long, at an angle of 45° with XY. Complete the hexagon. Above XY draw perpendiculars $7\frac{1}{2}''$ long immediately above the points b, c, d, e, and join the tops of these lines. This completes the elevation.

PROBLEM 12.

To project the same prism, lying with one face on H.P. ; *its longer edges to be inclined to the* V.P. *at an angle of* 30° ; *its nearest corner to be* $1\frac{1}{4}''$ *from* V.P. *Scale* $\frac{1}{6}$ *full size.* Fig. 22.

Fix the point f $1\frac{1}{4}''$ below XY. Draw the line fg, $7\frac{1}{2}''$ long, at an angle of 30° with XY ; and from f and g draw lines perpendicular to fg. From g, along the line gk, set off the distances g, m, l, k, equal to the distances c, i, p, o (Fig. 21). From the points, m, l, and k, draw lines parallel to fg till they meet the line fh. This completes the plan.

The heights K, H, G, which give the horizontal lines in the elevation, are obtained from the distances b, i, d in the plan (Fig. 21). Having obtained these heights, draw the lines $h'k'$ and $n'm'$. Carry up perpendicular lines from the points in the plan till they meet these lines, which give the corresponding points in the elevation. Join them as shown.

PROBLEM 13.

To project a regular hexagonal prism $12\frac{1}{2}''$ *long, with sides* 4″ *wide, resting on one of its smaller edges on the* H.P. ; *its longer edges to be inclined at an angle of* 60° *to the* H.P., *and parallel to the* V.P. *Its nearest edge to be* 2″ *from* V.P. *Scale* $\frac{1}{10}$ *full size.* Fig. 23.

In any convenient position draw the hexagon AEDF with 4″ sides, with lines joining the opposite angles, as shown. Draw XY, and at any point a' draw the line $a'c'$, $12\frac{1}{2}''$ long, at an angle of 60° with XY. From a' and c' draw the lines $a'b'$ and $c'd'$ perpendicular to $a'c'$. Set off the distances a', f', b' along $a'b'$, equal to the distances E, B, F of hexagon. From these points draw lines parallel to $a'c'$ till they meet the line $c'd'$. This completes the elevation.

Draw a line 2″ below XY, and parallel to it. From L on

this line let fall a perpendicular, and on it set off the distances L, K, H, G equal to the distances A, B, C, D of hexagon. From each of these points draw lines parallel to XY, and let fall lines from the various points in the elevation till they meet these lines. This gives the corresponding points in the plan. Join them as shown.

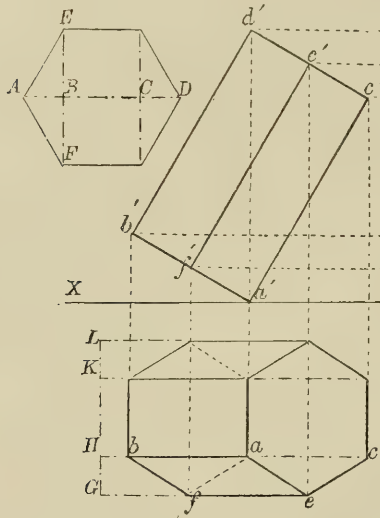


Fig. 23.

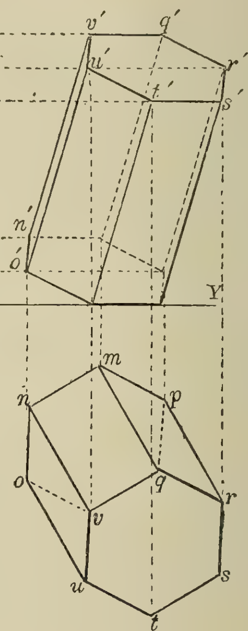


Fig. 24.

Problem 14.

To project the same prism standing on one of its shorter edges on the H.P., *at an angle of* 30° *with* V.P., *with its longer edges inclined at an angle of* 60° *with the* H.P., *and in vertical planes inclined at an angle of* 60° *to* V.P.; *its nearest corner to be* 3″ *from* V.P. *Scale* $\frac{1}{10}$ *full size.* Fig. 24.

Fix the position of point *m* 3″ below XY, and draw the line *mp* at an angle of 30° with XY. This represents the line *kh* in Fig. 23. The plan in Fig. 24 is precisely the same as

that shown in Fig. 23, turned to a different angle. Complete the plan against the line *mp*, from Fig. 23. As every point in the elevation is found in the same way, it is only necessary to describe one point. Erect a line on point *r* at right angles to XY till it meets a horizontal line drawn from point *e′*. This gives the point *r′*. Find the other points in the same way, and join them, as shown.

Problem 15.

To project a regular hexagonal prism, $7\frac{1}{4}''$ *long, with faces* $2\frac{1}{2}''$ *wide, resting on one corner on the* H.P.; *its longer edges to be inclined at an angle of* 45° *with the* H.P.; *one face to be parallel to the* V.P. *and* $2\frac{1}{2}''$ *from it. Scale* $\frac{1}{6}$ *full size.* Fig. 25.

Construct a hexagon with $2\frac{1}{2}''$ sides, and join the opposite angles by lines at right angles to each other, as shown.

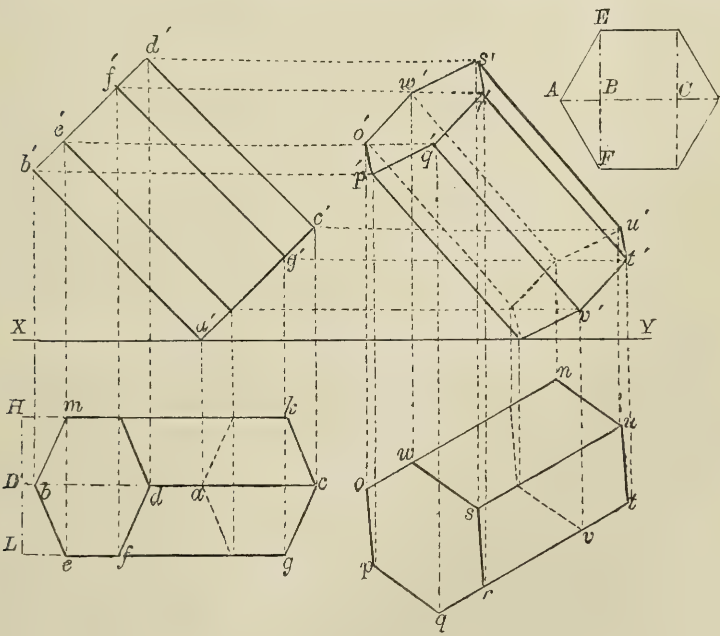


Fig 25. Fig. 26.

Draw the line XY, and at any point *a′* draw the line *a′b′*, $7\frac{1}{4}''$

long, at an angle of 45° with XY. From the points a' and b' draw the lines $a'c'$ and $b'd'$ perpendicular to $a'b'$. Set off the distances b', e', f', d on $b'd'$ equal to A, B, C, D of hexagon, and draw lines from these points parallel to $b'a'$ till they meet the line $a'c'$. This completes the elevation.

Draw the line mk $2\frac{1}{2}''$ below XY. From H on mk produced draw the line HL perpendicular to mk, and set off the distances H, D, L equal to the distances E, B, F of hexagon. Draw lines from these points parallel to XY. All the points of the plan must come on these three lines, and can easily be determined by dropping perpendiculars from the corresponding points in the elevation.

Problem 16.

To project the same solid, with its longer edges still inclined at an angle of 45° *to the* H.P. ; *but instead of being parallel to the* V.P., *as in the last problem, let them be in vertical planes, inclined at an angle of* 30° *with the* V.P. ; *its nearest corner to the* V.P. *to be* $1\frac{1}{4}''$ *from it. Scale* $\frac{1}{6}$ *full size.* Fig. 26.

Fix the position of point n $1\frac{1}{4}''$ below XY, and draw the line no at an angle of 30° with it. The plan in this problem is precisely similar to the plan in the last problem, but turned round at an angle of 30° with XY, and the line no corresponds with the line km (Fig. 25). Complete the plan as shown.

Every point in the elevation is found in the same manner. For example, erect a perpendicular on point o till it meets a horizontal line drawn from point e'; this will give the point o'. Proceed in the same way with all the other points, and join them.

The student should now pass on to Chapter XXIII., *Orthographic Projection,* before proceeding with the remaining problems in this subject.

CHAPTER XIX

THE REGULAR SOLIDS

THERE are five regular solids, and they are named after the number of faces they each possess; viz. the *Tetrahedron* has four faces, the *Hexahedron* six faces, the *Octahedron* eight faces, the *Dodecahedron* twelve faces, and the *Icosahedron* twenty faces.

They possess the following properties, viz.:—

The faces of each solid are equal, and similar in shape, and its edges are of equal length.

All their faces are regular polygons.

All the angles formed by the contiguous faces of each solid are equal.

If they were each inscribed in a sphere, all their angular points would be equidistant from its centre.

PROBLEM 17.

To project a tetrahedron with edges 9″ long, with one face resting on the H.P.; *one of its edges to be at an angle of* 16° *with* V.P., *and its nearest angular point* $3\frac{3}{4}''$ *from* V.P. *Scale* $\frac{1}{8}$ *full size.* Fig. 27.

All the faces of this solid are equilateral triangles.

Draw XY, and fix the position of point *a* $3\frac{3}{4}''$ below it. From *a* draw the line *ab* 9″ long, at an angle of 16° with XY. On the line *ab* construct an equilateral triangle *abc*. Bisect each of the angles at *a*, *b*, and *c* by lines meeting at *d*. This completes the plan.

To find the altitude of the elevation, produce the line *bd* to *e*, and at *d* draw the line *df* perpendicular to *eb*. With *b* as centre, and radius *bc*, draw an arc cutting *df* in *f*.

To draw the elevation, erect a perpendicular *d′g′* directly above *d*, and make *d′g′* equal to *df*. Carry up the points *a, c, b* till they meet XY in *a′*, *c′*, and *b′*. Join the point *d′* to *a′*, *c′*, and *b′*.

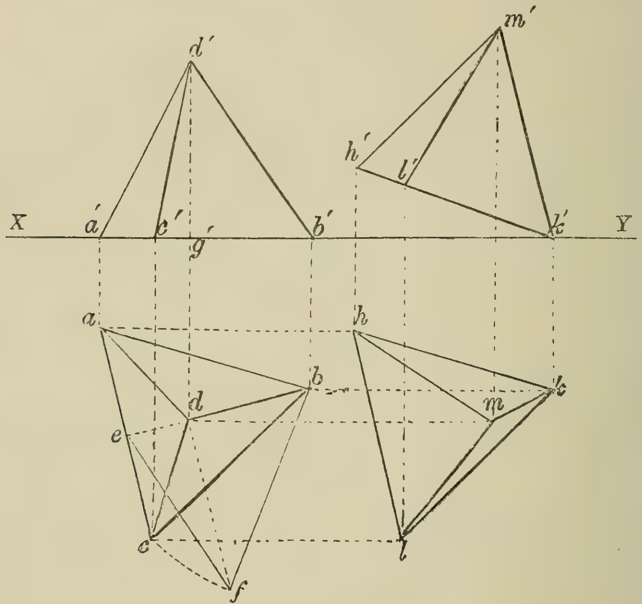


Fig. 27. Fig. 28.

PROBLEM 18.

To project the same solid tilted on to one of its angular points with its base inclined at an angle of 20° *with* H.P. *Scale* $\frac{1}{8}$ *full size.* Fig. 28.

Fix the point *k′* on XY, and draw the line *k′h′* at an angle of 20° with it, and equal in length to the line *b′a′* (Fig. 27). As this elevation is precisely similar to the last, complete it from that figure on the line *h′k′*.

Draw lines from the points *a, b, d,* and *c* (Fig. 27) parallel

to XY. All the points of the plan must come on these four lines, and are found by dropping lines at right angles to XY from the corresponding points in the elevation.

Problem 19.

To project a hexahedron or cube with edges $3\frac{3}{4}''$ long, resting on one edge on the H.P., *and its base making an angle of* 22° *with it; the side nearest the* V.P. *to be parallel to, and $1\frac{3}{4}''$ from it. Scale $\frac{1}{4}$ full size.* Fig. 29.

All the faces of this solid are squares.

Draw the line XY, and at any point *a′* draw the line *a′b′* at an angle of 22°. Complete the square *a′b′c′d′*.

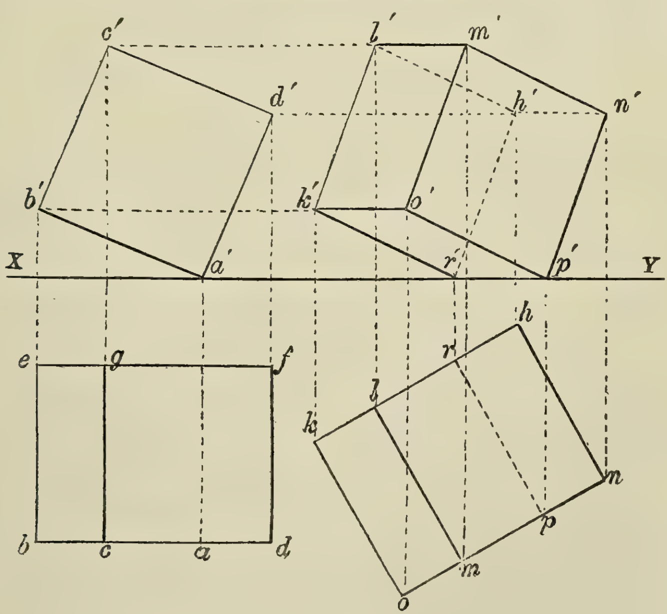


Fig. 29. Fig. 30.

Below XY draw the line *ef* $1\frac{3}{4}''$ from it. Draw the lines *b′b*, *cc′* and *d′d* at right angles to XY. Make *eb* and *fd* each equal to *a′b′*, and join *bd*; *cc′* cuts *ef* in *g*. Join *cg*.

Problem 20.

To project the same solid with its edge still resting on the H.P. *and forming the same angle with it, but inclined to the* V.P. *at an angle of* 30°*; its nearest corner to be* 1″ *from* V.P. *Scale* ¼ *full size.* Fig. 30.

Fix the position of point *h* 1″ below XY, and draw the line *hk* equal to *fe* (Fig. 29) at an angle of 30° with XY. Complete this plan from Fig. 29, which is precisely similar in shape.

Draw lines through the points *b′*, *c′*, and *d′* (Fig. 29) parallel to XY. Draw perpendiculars from the points in the plan till they meet these lines, which give the corresponding points in the elevation. Join these points as shown.

Problem 21.

To project an octahedron, with edges 4″ *long, poised on one of its angular points on the* H.P.; *its axis to be perpendicular to* H.P., *and its edge nearest to the* V.P. *to be parallel to, and* 1¾″ *from it. Scale* ¼ *full size.* Fig. 31.

All the faces of this solid are equilateral triangles.

Draw the line XY, and 1¾″ below it draw the line *ab* 4″ long. Complete the square *abcd,* and draw its diagonals. This will be the plan of the solid.

As every angular point is equidistant from the centre of a sphere circumscribing the solid, the point *e* must be the centre of the plan of the sphere and *ad* its diameter.

Draw the projector *ee′* from the point *e*. Make *e′f′* equal in length to *ad*. This will be the axis of the solid. Bisect *e′f′* by the line *c′d′* parallel to XY, and erect perpendiculars from the points *c* and *d* till they meet this line in the points *c′* and *d′*. Join *c′e′* and *c′f′*, *d′e′* and *d′f′*, to complete the elevation.

Problem 22.

To project the same solid with one face resting on the H.P.; *the edge nearest the* V.P. *to be parallel to, and* $1\frac{3}{4}''$ *from it. Scale* $\frac{1}{4}$ *full size.* Fig. 32.

Fix the positions of the points g' and h' on XY the same

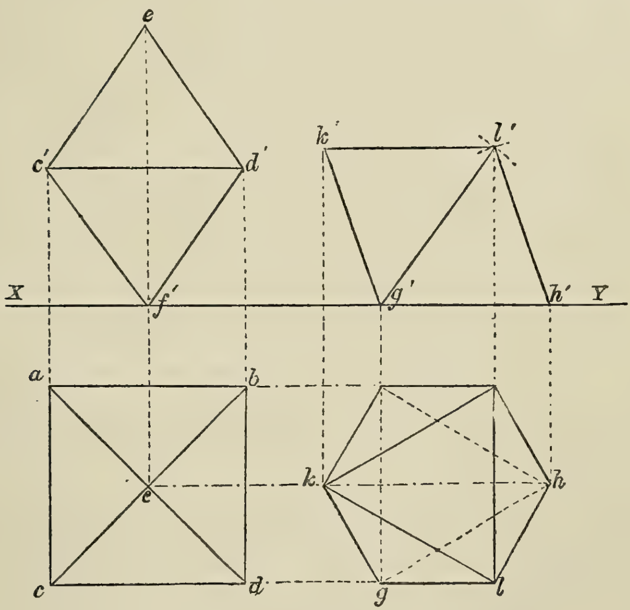


Fig. 31. Fig. 32.

distance apart as the points c' and f' in the preceding problem. On this line complete the elevation from Fig. 31.

Draw lines from the points b, e, and d (Fig. 31) parallel to XY. All the points of the plan must come on these three lines, and are found by dropping perpendiculars from their corresponding points in the elevation.

Problem 23.

To project a dodecahedron with edges $3\frac{1}{2}''$ long, with one face resting on the H.P., *one edge of which is parallel to* V.P. *and $4\frac{1}{2}''$ from it. Scale $\frac{1}{6}$ full size.* Fig. 33.

Each side of this solid is a regular pentagon, with opposite faces parallel to each other, and their angles alternating.

Draw the line XY, and $4\frac{1}{2}''$ below it draw the line *ab* $3\frac{1}{2}''$ long. This will be one edge of the pentagon forming the base of the solid; on the line *ab* complete the pentagon. Draw another equal pentagon on this, with alternating angles, to represent the top face of the solid. To get the position of point *w* we will imagine another face of the solid hinged on the line *ac*, and lying flat on the H.P.; so construct the pentagon *ac*UVW against the line *ac*. This face of the solid is said to be "constructed" on to the H.P. Also construct one side of a similar pentagon *a*W′ in connection with the line *ab*.

It is evident that the distance between the points *u* and *w* must be the same as between U and W, so draw lines from U and W perpendicular to the line *ca;* also from the point W′ perpendicular to the line *ab.* Where these lines from W and W′ intersect each other give the point *w.* Find the centre O of the pentagon *abcde;* and with this point O as centre, and radius O*w*, describe a circle. From the centre O draw lines through the angles of the pentagons forming the base and top of the solid till they meet the circle in the points *m, n, p, q, r, s, t, u, v,* and *w,* by joining which the plan is completed.

In the elevation all the angular points of the solid are at four distinct levels, viz. X, U″, V‴, and L′. To obtain these levels we must find the angle the pentagon *ac*UVW (on plan) will form with the H.P. when it is rotated on the line *ac* into its proper position. We know that *c*U is the real length of the line *cu,* and that its perpendicular distance from *ca* is CU;

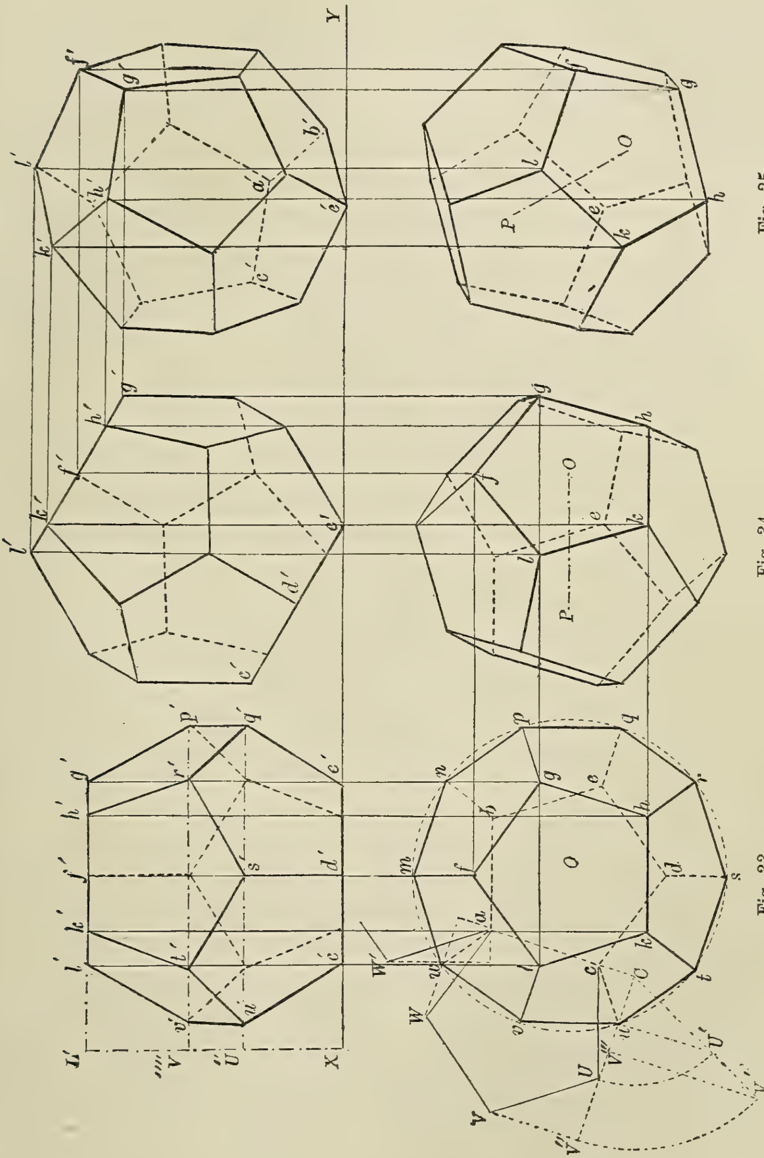


Fig. 33.

Fig. 34.

Fig. 35.

so take C as centre, and radius CU, and draw an arc UU′ till it meets the line uU′ drawn perpendicularly to UC. Join CU′. With C as centre, and radius CV″, describe another arc till it meets CU′ produced in V′. Then V′CV″ is the angle required. Consider V″C as the level of the H.P., then the line uU′ is the perpendicular height of XU″ in elevation, and V‴V′ the height of XV⁗. To get the level of the highest series of points in elevation, make V⁗L′ equal to XU″. Draw lines through the points U″, V⁗, and L′ parallel to XY. To obtain the points of the solid in elevation, erect perpendiculars from corresponding points in plan till they meet these lines.

To save unnecessary complication and confusion, one face only of the solid (the top one) is connected with plan by projectors. All the other points are found in the same way.

Problem 24.

To project the same solid in the same position relative to the V.P., *but tilted on one of its angular points at an angle of* 30° *with* H.P. *Scale* $\frac{1}{6}$ *full size.* Fig. 34.

Fix the point e' on XY, and draw the line $e'c'$ at an angle of 30° with it. On this line complete the elevation from that of the preceding figure.

To draw the plan, let fall lines from the various points of the elevation, at right angles to XY, till they meet lines drawn parallel to XY from the corresponding points in the plan of Fig. 33.

To avoid confusion, the uppermost face of the solid only is taken, and the same letters are retained throughout, both as regards the preceding problem as well as the one following; by this means the same points can be recognised throughout their various positions.

The axis of this solid is determined by drawing projectors from the points f' and d' till they meet a line drawn from O (Fig. 33) in the points O and P.

PROBLEM 25.

To project the same solid, poised on the same angular point, and forming the same angle with the H.P., *but turned round so that its axis* OP *is in a vertical plane at an angle of* 60° *with* V.P., *instead of being parallel to it as in the last problem. Scale $\frac{1}{6}$ full size.* Fig. 35.

The point *e* remains unchanged in position throughout the three problems, as it forms the pivot on which the object is turned; so fix the position of *e* the same distance below XY as it is shown in Fig. 34. Draw the axis OP the same distance from point *e* as there shown; only it must be at an angle of 60° with XY, instead of parallel to it as in Fig. 34.

As the plan is precisely similar to Fig. 34 turned round to a different angle, complete the construction from that figure.

To draw the elevation, erect lines from the various points in the plan, at right angles to XY, till they meet lines drawn from the corresponding points in the elevation of Fig. 34, parallel to XY.

The connecting lines for one face only of the solid is shown, to avoid confusion.

PROBLEM 26.

To project an icosahedron with edges 6″ *long; one face to rest on the* H.P., *with one of the edges forming the base at an angle of* 45° *with* V.P., *and the corner of the base nearest the* V.P. *to be* $5\frac{3}{4}''$ *from it. Scale $\frac{1}{8}$ full size.* Fig. 36.

> This solid is composed of twenty equilateral triangles; its opposite faces are parallel to each other, with their angles alternating.

Draw the line XY, and $5\frac{3}{4}''$ below it fix the point *a*. Draw the line *ab* 6″ long, and at an angle of 45° with V.P. On *ab* construct the equilateral triangle *abc*. This represents the

plan of the base. On this base, with alternating angles, draw an equal and similar triangle *def*; this represents the top face of the solid. If we analyse the solid we shall find that these two triangles are connected by shallow pentagonal pyramids, and by

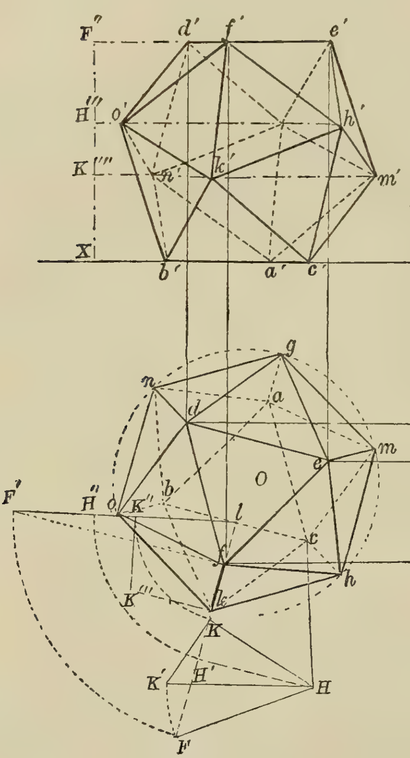


Fig. 36.

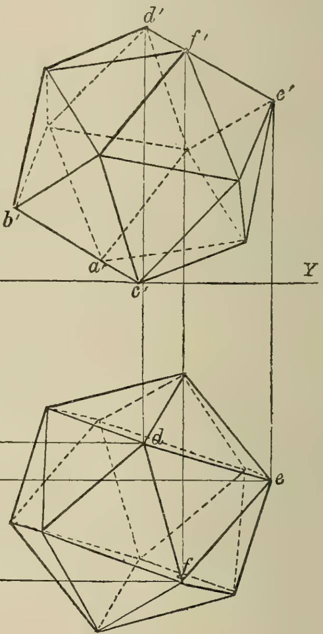


Fig. 37.

projecting one of these pyramids we shall be enabled to complete the solid.

On the line *bc* construct two sides of a regular pentagon *c*H and HF. These represent the lines *ch* and *hf*, forming part of the base of the pentagonal pyramid constructed on to the H.P. To find the angle this base forms with the H.P. Bisect the line *bc* in *l*. Join *l*F. With *l* as centre, and *l*F as radius,

draw an arc till it meets a line drawn from f parallel to bc in F′. Join Fl. Then FlF′ is the angle required. To find the position of the vertex k. Find the centre K of the constructed pentagon, and join KH. From K draw KK′ perpendicular to KH. With H as centre, and radius equal to the length of one edge of the solid (6″), draw an arc intersecting KK′ in K′. With l as centre, and lK radius, draw an arc cutting lF′ in K″. From K″ draw a line K″K‴ perpendicular to lF′, and equal to KK′ in length; and from K‴ draw a line K‴k parallel to bc till it meets lF in k. This gives the position of vertex k.

Find the centre O of the triangle abc. With centre O, and radius Ok, draw a circle; this is the plan of the circumscribing sphere. Draw lines radiating from O through the angles of the two triangles abc and def till they meet the circle in the points k, o, n, g, m, and h. Join these points; also join them with the angles of the two triangles abc and def. This completes the plan.

The points in the elevation all come at four distinct levels, viz. X, K⁗, H‴, and F″. If we take the line Fl on plan to represent the level of the H.P., FlF′ will represent the angle the base of the pentagonal pyramid forms with it; so the line fF′ is the perpendicular height of F″ above XY. In the same way the height of K⁗ is found by taking the perpendicular height of K‴ above the line Fl. To obtain the height of H‴, draw a line from H (on plan) perpendicular to Fl till it meets Fl in H′. With l as centre, and lH′ as radius, draw an arc till it meets lF′ in H″. The perpendicular height of this point above the line Fl gives the height of H‴ above X.

From K⁗, H‴, and F″ draw lines parallel to XY; and to complete the elevation draw perpendicular lines from the corresponding points in the plan till they meet these lines, and join them.

The projectors to one face only are shown (the top face), to save confusion, but the angular points bear corresponding letters to those on plan.

Problem 27.

To project the same solid, poised on one of its angular points, at an angle of 30° *with* H.P., *but bearing the same relation to* V.P. *as in the preceding problem. Scale* $\frac{1}{8}$ *full size.* Fig. 37.

Fix the position of point c' on XY, and draw the line $c'b'$ at an angle of 30° with it. Complete the elevation on $c'b'$ from the elevation Fig. 36, which is precisely similar.

Let fall lines from the various points in the elevation, at right angles to XY, till they meet lines drawn from the corresponding points in plan (Fig. 36) parallel to XY. Only one face is shown thus connected (the top face), to save confusion of lines, but the points bear corresponding letters throughout both problems.

Problem 28.

To find the relative lengths of the edges of the five regular solids. Fig. 38.

Let AB be the diameter of the circumscribing sphere.

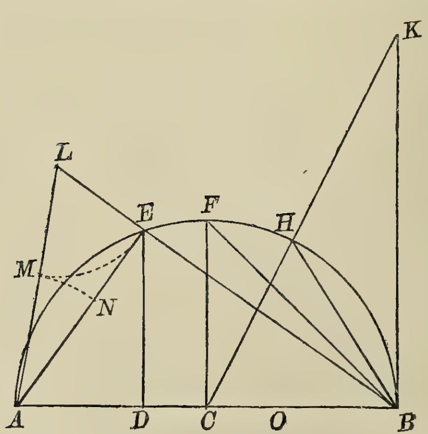


Fig. 38.

Bisect AB in C. With C as centre, and CA as radius, draw

the semicircle AFB. Trisect AB in D and O. Draw DE perpendicular to AB, and join EB, which gives the edge of the tetrahedron.

Join EA, which gives the edge of the *hexahedron*.

At C draw CF perpendicular to AB, and join FB, which gives the edge of the *octahedron*.

At E draw EL perpendicular to AE, and half its length. With L as centre, and LE as radius, draw an arc cutting LA in M. With A as centre, and AM as radius, draw an arc cutting AE in N. AE will then be medially divided in N, and AN gives the edge of the *dodecahedron*.

At B draw BK perpendicular to AB, and equal to it in length. Join KC by a line cutting the semicircle in H. Join HB, which gives the edge of the *icosahedron*.

$$AB : EB :: \sqrt{3} : \sqrt{2}.$$
$$AB : AE :: \sqrt{3} : 1.$$
$$AB : FB :: \sqrt{2} : 1.$$

Take a line HB equal to HB (Fig. 38), and construct an equilateral triangle upon it (Fig. 39); describe a circle about it; and inscribe a regular pentagon in the same circle, then each side of the pentagon is equal to the line AN (Fig. 38). From this the student will see that the opposite faces of the dodecahedron and icosahedron are the same distance apart when they are both circumscribed by a sphere of the same diameter.

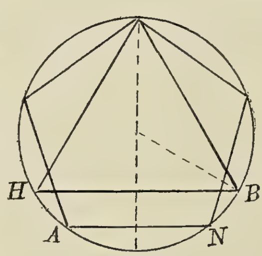


Fig. 39.

CHAPTER XX

OCTAGONAL PYRAMIDS, CONES, AND CYLINDERS

OCTAGONAL PYRAMIDS.

PROBLEM 29.

To project a regular octagonal pyramid, 8″ high, with each side of its base $2\frac{1}{2}''$ *wide, standing on its base on the* H.P., *with an edge of its base parallel to the* V.P. *and* $2\frac{1}{2}''$ *from it. Scale* $\frac{1}{8}$ *full size.* Fig. 40.

Draw XY, and $2\frac{1}{2}''$ below it draw *ab* $2\frac{1}{2}''$ long. On *ab* construct a regular octagon, and join the opposite angles. Carry up projectors from the points *d, e, f, g* perpendicular to XY. Fix the point c' immediately above *c*, and 8″ above XY. Join the point c' to d', e', f' and g', which completes the pyramid.

PROBLEM 30.

To project the same solid lying with one face on the H.P., *but with its axis in a plane parallel to the* V.P. *Scale* $\frac{1}{8}$ *full size.* Fig. 41.

On XY mark off the distance $h'k'$ equal to $g'c'$ (Fig. 40). Complete the construction of elevation from Fig. 40.

Let fall lines from the various points of the elevation, at right angles to XY, till they meet lines drawn from the corresponding points in the plan (Fig. 40), parallel to XY. The intersection of

these lines give the required points, by joining which we obtain the plan.

Problem 31.

To project the same solid resting on one of its shorter edges, with its base inclined at an angle of 30° *with* H.P. ; *its axis to be in a plane parallel to the* V.P. *Scale* $\frac{1}{8}$ *full size.* Fig. 42.

Draw XY, and fix the position of point o'. Draw $o'p'$ at an

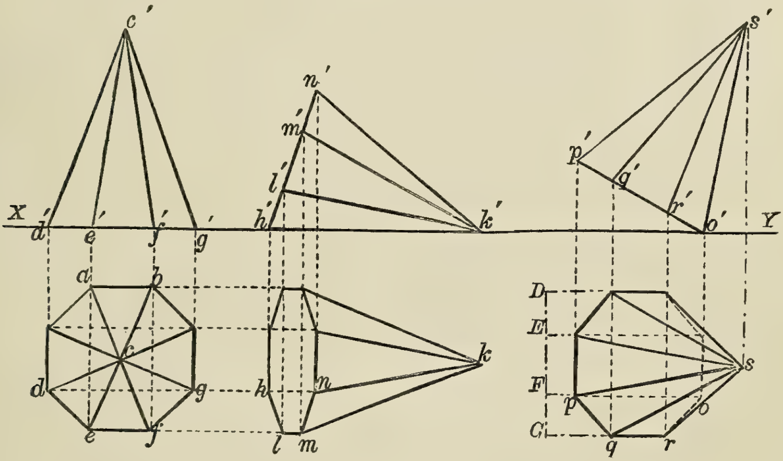


Fig. 40. Fig. 41. Fig. 42.

angle of 30° with XY, and equal in length to $h'n'$ (Fig. 41). Complete the elevation from Fig. 41.

In any convenient position draw the line DG perpendicular to XY, and set off upon it the distances D, E, F, G equal to h', l', m', n' (Fig. 41). Draw lines from these points parallel to XY. From the various points in the elevation let fall lines at right angles to XY till they meet these lines, which give the corresponding points in the plan.

CONES.

Cones are projected in precisely the same way as polygonal pyramids. After finding the points in the base, instead of

joining them by lines, as in the case of pyramids, a fair curve is drawn through them. Eight points of the base are found in the examples here given: a number deemed sufficient for ordinary purposes. Should more points be required, it is only necessary to select a pyramid having more sides than eight to construct the cone upon.

Problem 32.

To project a cone 8″ high, with base $6\frac{1}{2}$″ in diameter, resting on the H.P. *Scale $\frac{1}{8}$ full size.* Fig. 43.

Draw XY, and in any convenient position below it draw a circle $6\frac{1}{2}$″ in diameter.

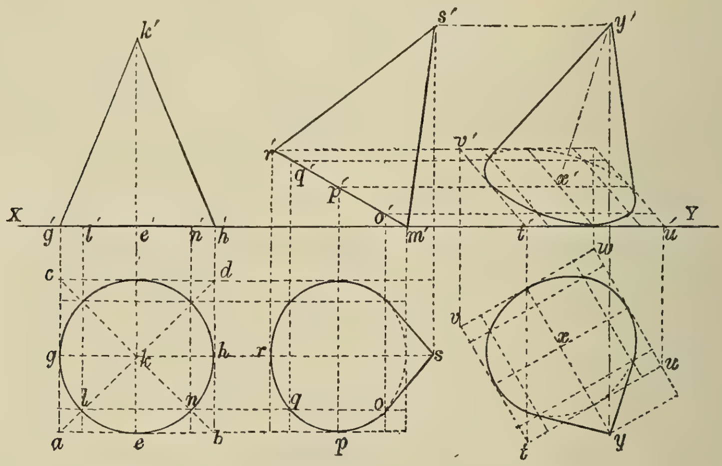


Fig. 43. Fig. 44. Fig. 45.

Carry up projectors from g and h till they meet XY; also from k, and produce the same 8″ above XY. Join $k'g'$ and $k'h'$.

Problem 33.

To project the same solid resting on its edge, with its base inclined at an angle of 30° *with* H.P.; *its axis to be in a vertical plane parallel to the* V.P. *Scale $\frac{1}{8}$ full size.* Fig. 44.

Inscribe the circle (Fig. 43) in a square *abcd*, the side nearest

XY to be parallel to it. Draw diagonals, and through the centre draw diameters parallel to the sides of the square. Through the four points where the diagonals cut the circle draw lines parallel to the sides of the square.

Fix the point m' on XY, and draw the line $m'r'$ at an angle of 30° with it. Set off the distances $m'o'p'q'r'$ equal to the distances h', n', e', l', g' (Fig. 43). Complete the elevation from Fig. 43.

Let fall lines from the various points of the elevation at right angles to XY till they meet lines drawn from the corresponding points of the plan (Fig. 43). Draw a fair curve through the points forming the base, and lines from the vertex s tangential to the base. This completes the plan.

Problem 34.

To project the same solid, resting on its edge, with its base still inclined at an angle of 30° *with* H.P., *but with its axis in a vertical plane, inclined at an angle of* 60° *with* V.P. *Scale* $\frac{1}{8}$ *full size.* Fig. 45.

Draw the lines enclosing the base with the parallel lines intersecting each other where the diagonals cut the circle from the plan (Fig. 44). The line xy that passes through the axis to be inclined at an angle of 60° with XY. Complete the plan from Fig. 44.

Draw lines from the various points of the plan at right angles to XY till they meet lines drawn from the corresponding points of the elevation (Fig. 44). Complete the elevation as shown.

CYLINDERS.

In the examples here given, only eight points of the circular base are projected, to save confusion of lines; but any number of points can be found in the same manner.

Problem 35.

To project a cylinder, $5\frac{1}{2}''$ in diameter and $8\frac{1}{4}''$ high, standing on its base on the H.P. *Scale $\frac{1}{6}$ full size.* Fig. 46.

Draw XY, and in any convenient position below it draw a circle $5\frac{1}{2}''$ in diameter. Draw tangents to it at the points *a* and

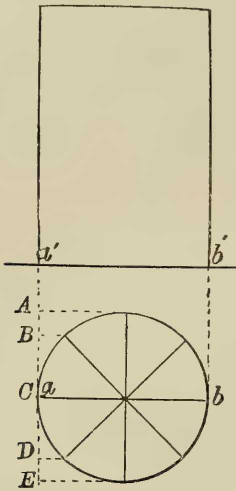


Fig. 46.

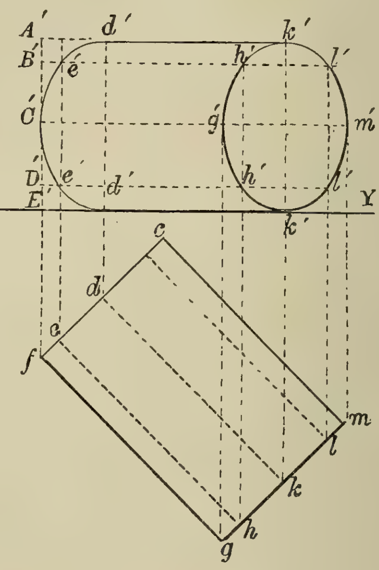


Fig. 47.

b perpendicular to XY, and produce them $8\frac{1}{4}''$ above XY. Join the tops of these lines to complete the cylinder.

Problem 36.

To project the same cylinder, lying on its side on the H.P., *with its axis inclined at an angle of* 45° *with the* V.P. *Scale $\frac{1}{6}$ full size.* Fig. 47.

Draw four diameters to the plan (Fig. 46), by first drawing the line *ab* parallel to XY, and then the other three diameters

equidistant from it. This gives eight points in the circumference. Draw lines from these points parallel to XY, till they meet the line AE in the points A, B, C, D, E.

At any convenient point *c* below XY (Fig. 47), draw the line *cf* at an angle of 45° with it. Draw the line *cm*, $8\frac{1}{4}''$ long, perpendicular to *cf*, and from *m* draw *mg* parallel to *cf*. From *m*, along *mg*, set off the distances *m, l, k, h, g* equal to the distances A, B, C, D, E (Fig. 46), and from these points draw lines parallel to *cm*. This completes the plan.

Draw a line from *f* perpendicular to XY, and produce it above the ground line. Set off the distances E′, D′, C′, B′, A′ on this line, equal to the distances E, D, C, B, A (Fig. 46), and draw lines from these points parallel to XY. Draw the projectors from the various points in the plan till they meet these lines, which give the projections of the points required to complete the elevation.

Problem 37.

To project the same solid, resting on its edge, with its base inclined at an angle of 30° *with the* H.P., *its axis being parallel to the* V.P. *Scale $\frac{1}{6}$ full size.* Fig. 48.

Draw XY, and at any point *n*′ upon it draw the line *n*′*r*′ at an angle of 30° with the ground line. On the line *n*′*r*′ set off the distances *n*′, *o*′, *p*′, *q*′, *r*′ equal to the distances *g*′, *h*′, *k*′, *l*′, *m*′ (Fig. 47), and from each of these points draw perpendiculars to *n*′*r*′, $8\frac{1}{4}''$ long. Join the tops. This completes the elevation.

From *r*′ let fall a line at right angles to XY, and set off upon it from any convenient point A the points B, C, D, E equal to the distances *n*′, *o*′, *p*′, *q*′, *r*′. From each of these points draw lines parallel to XY. From the various points in the elevation drop projectors till they meet these lines in the corresponding points, by connecting which we get the plan.

Problem 38.

To project the same solid resting on its edge, with its base still inclined at an angle of 30° *with the* H.P., *but with its axis inclined at an angle of* 60° *with the* V.P. *Scale* $\frac{1}{6}$ *full size.* Fig. 49.

Draw the line FG equal to AE (Fig 48), inclined at an angle

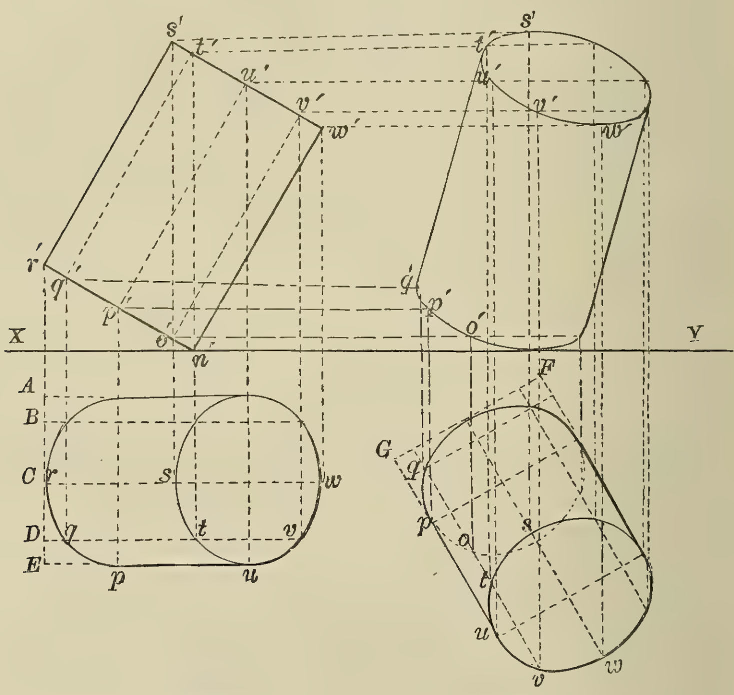


Fig. 48. Fig. 49.

of 30° with XY. Complete the plan from Fig. 48. Draw projectors from the plan till they meet lines drawn parallel to XY from the corresponding points in the elevation (Fig. 48). These give the necessary projections for completing the elevation.

CHAPTER XXI

SPHERES, GREAT CIRCLES, SPHERICAL TRIANGLES, AND HELICES

SPHERES.

THE plan and elevation of a sphere are simple circles; but if we divide the sphere into divisions by lines upon its surface, such as meridians of longitude and parallels of latitude, we shall be enabled to fix its position and inclination to the co-ordinate planes, and project it accordingly.

So as not to confuse the figure too much, we will restrict ourselves to eight meridians, with the equator, and two parallels of latitude. The junction of the meridians will of course give us the position of the poles, which will determine the axis.

PROBLEM 39.

To project a sphere $5\frac{1}{2}''$ in diameter, with meridians and parallels; its axis to be perpendicular to the H.P. *Scale $\frac{1}{4}$ full size.* Fig. 50.

Draw XY, and in any convenient position below it draw a circle $5\frac{1}{2}''$ in diameter. Draw the diameters ab parallel to the ground line, and dt at right angles to it, and two other diameters equidistant from them.

Produce the line dt above XY, and make $c'p'$ equal in length to the diameter ab. Bisect $c'p'$ in d'. With d' as centre, and radius equal to pa, draw a circle. Draw $a'b'$ through d' till it

meets the circle in a' and b'. Through d' draw the line $e'f'$ at an angle of 45° with $a'b'$, till it meets the circle in e' and f'. From e' and f' draw lines parallel to $a'b'$ till they meet the circle in g' and h'. These lines represent parallels of latitude. Drop a perpendicular from g' till it meets ab in g. With p as

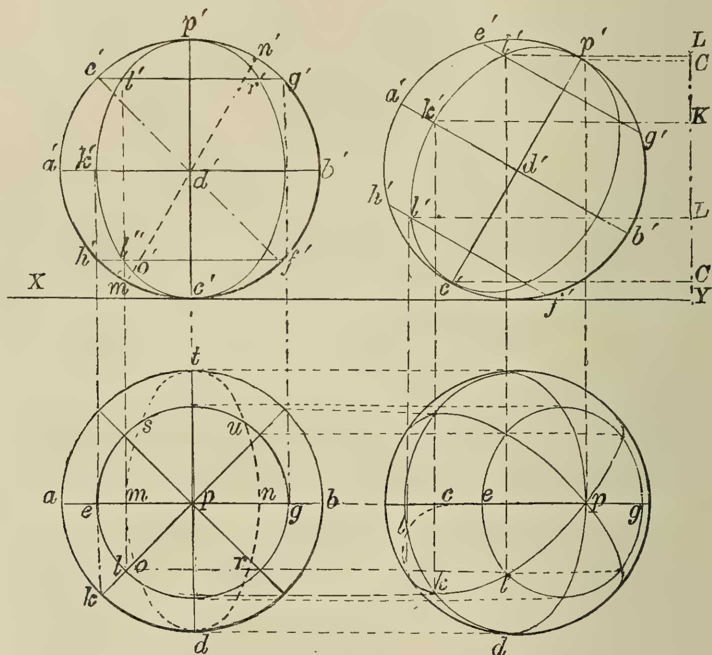


Fig. 50. Fig. 51.

centre, and radius pg, draw a circle. This is the plan of the parallel $e'g'$.

To avoid confusion, the projectors for half of one meridian only are shown ; but they are all found in the same manner.

Erect a perpendicular on point k till it meets the equator in point k' ; also from point l till it meets the parallels in points l'' and l'. Draw a curve through the points c', l'', k', l', p', which gives the projection of the meridian.

Problem 40.

To project the same sphere, with its axis inclined to the H.P. *at an angle of* 60°, *but parallel to the* V.P. *Scale* $\frac{1}{4}$ *full size.* Fig. 51.

Note.—The same letters are taken throughout these spherical problems to facilitate reference.

Draw a circle $5\frac{1}{2}''$ in diameter, resting on XY, and draw the line $c'p'$ at an angle of 60° with it. Draw the line $a'b'$ at right angles to $c'p'$, and set off the distances of the parallels above and below $a'b'$ equal to their distances in the elevation (Fig. 50). Draw the lines $e'g'$ and $h'f'$ parallel to $a'b'$. Complete the elevation from Fig. 50. Draw lines from all the points of intersection between the meridians and parallels of the plan (Fig. 50) parallel to XY, and let fall perpendiculars from the corresponding points in the elevation till they meet these lines, which give the projections of the points of intersection. Draw the curves.

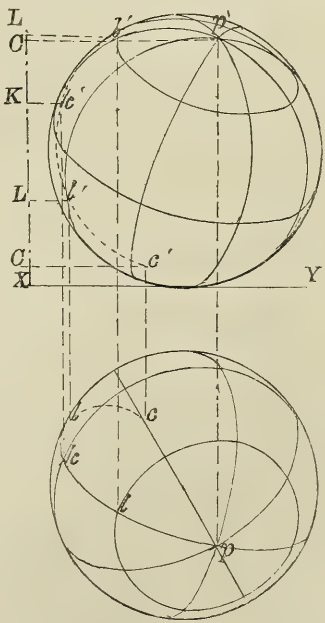


Fig. 52.

Problem 41.

To project the same sphere, with its axis still inclined to the H.P., *at an angle of* 60°, *but in a vertical plane inclined at an angle of* 60° *with the* V.P. *Scale* $\frac{1}{4}$ *full size.* Fig. 52.

Draw the line cp at an angle of 60° with XY, for the plan of the axis, and on this line complete the plan from Fig. 51. Perpendicular to XY draw the line XL, and set off the distances X, C, L, K, C, L equal to the distances Y, C, L, K, C, L (Fig. 51).

From each of these points draw lines parallel to XY till they meet projectors drawn from the corresponding points in the plan, which give the projections required.

GREAT CIRCLES.

If a sphere is intersected by a plane, the section is always a circle. When the plane passes through the centre of the sphere, the section is called a *great circle; e.g.* the equator and also the meridians are great circles.

Let us take the sphere (Fig. 50) and construct upon it the following problem.

Problem 42.

To project a great circle upon a given sphere, inclined at an angle of 60° *to the plane of the equator.* Fig. 50.

Draw the line $m'n'$ in the elevation, at an angle of 60° to the line $a'b'$. This line will cut the projection of the equator in the point d', and the parallels in o' and r'. Let fall projectors from these points till they meet the corresponding circles in the plan; these give the following projections, viz. n, r, d, o, m, s, t, and u. By drawing a fair curve through these points we obtain a plan of the great circle in the required position.

Note.—The projectors are not drawn in the figure, to avoid confusion.

All the meridians we have projected in Figs. 50, 51, and 52 are projections of great circles. By studying these, the student will have no difficulty in projecting great circles upon a sphere in any desired position.

SPHERICAL TRIANGLES.

A *spherical triangle* is formed by the intersection of the planes of three great circles with the surface of a sphere. These three planes form what is called a *trihedral angle* (one of the angular corners of the tetrahedron), the vertex of which

must always be the centre of the sphere. The three lines which form the vertex are radii of the sphere, and are called the *edges* of the trihedral angle. A similar trihedral angle is always subtended on the opposite side of the centre.

PROJECTION OF SPHERICAL TRIANGLES.

A trihedral angle is formed at O′ (Fig. 53), and the lines O′a', O′b', and O′c' are its edges.

The angles of the spherical triangle are formed at the points

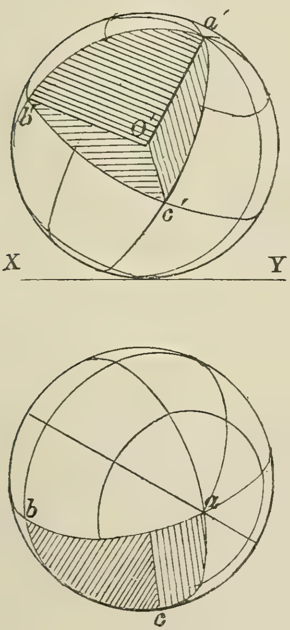


Fig. 53.

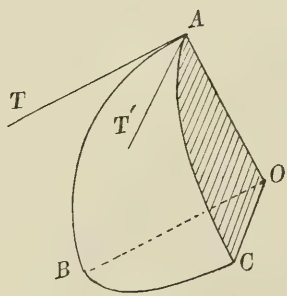


Fig. 54.

a', b', c' (Fig. 53) by the sides of the triangle, and are the angles which the planes of the sides form with one another. They may be measured by the tangents, as at TAT′ (Fig. 54), or by the arc BC cut off by the sides when produced to be quadrants.

As we can project great circles at any required inclination to

a sphere, we shall have little difficulty in projecting spherical triangles.

SOLUTION OF SPHERICAL TRIANGLES.

The problems in this subject appear rather complicated, on account of the number of angles contained in each problem, and the consequent number of lines necessary for their construction; but if a simple paper model were constructed, the principles illustrated would be more readily understood. This can be done in the following manner:—

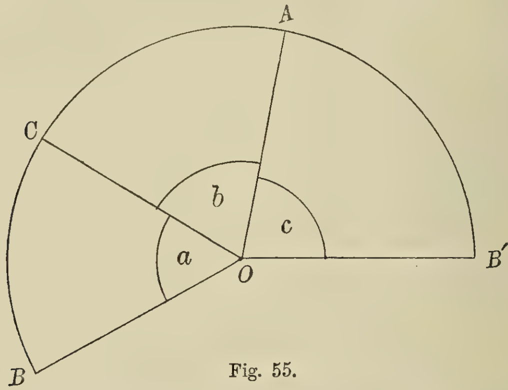


Fig. 55.

Cut out the sector of a circle (Fig. 55) of any radius, and then fold it along any two radii OC and OA till the free edges OB and OB′ coincide. This model is represented in Fig. 56.

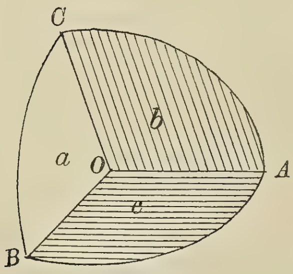


Fig. 56.

The three sides of the spherical triangle are represented by the letters a, b, and c, and the angles formed by the three sides by the letters A, B, and C. The vertex of the trihedral angle is O.

C is the angle between a and b.

A	„	„	b	„	c.
B	„	„	c	„	a.

The plane of the great circle which forms the *equator* has an imaginary line passing through its centre at right angles to it, and terminated at the *poles* on the surface of the sphere; this line is called the *polar axis*. Every great circle has an axis.

In Fig. 57 the planes of three great circles are shown intersecting each other so as to form the spherical triangle ABC. If we take lines perpendicular to the outer surfaces of the

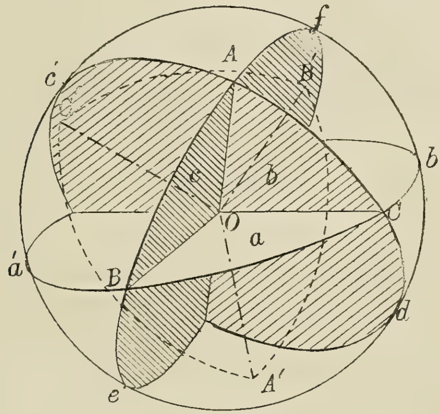


Fig. 57.

planes forming this spherical triangle,—for instance, OA′ is perpendicular to the plane *a′b′*, OB′ to *c′d*, and OC′ to *ef*,—their poles will meet the surface of the sphere in the points A′, B′, and C′. If these points are connected by arcs of great circles, they will form another spherical triangle A′B′C′, *the sides of which are supplementary to the angles of the triangle* ABC, *and vice versa; e.g.*:—

If the side a is 60°, the angle A′ will be $180° - 60° = 120°$.
If the side b is 70°, the angle B′ will be $180° - 70° = 110°$.
If the side c is 80°, the angle C′ will be $180° - 80° = 100°$.

If we were to draw axes to the inner surfaces of the planes forming the triangle ABC, and join their extremities by arcs of great circles, another spherical triangle would be formed precisely

similar to A′B′C′, but with its angles reversed in position, and it would be on the same side of the sphere as ABC.

This triangle is called the *polar triangle;* ABC being called the *primitive triangle.*

Note.—A′B′C′ is shown (Fig. 57) on the opposite side of the sphere to ABC, to make the illustration clearer; it is really a polar triangle, but there is a convention by which the polar triangle is always taken on the same side of the sphere as the primitive triangle.

If we were to take A′B′C′ to represent a primitive triangle, then ABC would be the polar triangle. Each triangle forms a trihedral angle at the centre of the sphere and they are mutually polar to each other; for instance, the line OC would be perpendicular to a plane containing the arc A′B′, and so on with the other lines.

A convention is entered into that no side of a spherical triangle should be greater than a semicircle.

The following properties of these triangles should be borne in mind, viz.:—

The sum of any two sides will be greater than the third.
The sum of the three sides cannot exceed four right angles.
The sum of the angles A, B, and C will exceed two, and be less than six right angles.
A, B, and C must each be less than two right angles.

There are three sides—a, b, and c—and three angles—A, B, and C—to each spherical triangle, from any three of which it can be solved; we can therefore have six variations in the necessary data (Fig. 56).

1. If the sides a, b, and c are given, A, B, and C can be determined.
2. If two sides and the angle between them are given, *e.g.* a, b, and C, we can obtain A, B, and c.
3. If two sides and the angle opposite one of them are

given, for instance a, b, and A, then we can determine B, C, and c.

4. If one side and its two adjacent angles are given, *i.e.* a, B, and C, then A, b, and c can be obtained.
5. If one side and its opposite and adjacent angles are given, for instance a, A, and C, we can determine b, B, and c.
6. If the three angles A, B, and C are given, the three sides a, b, and c can be found.

Note.—In the following problems the given sides are indicated by continuous arcs, and those required by dotted arcs.

Problem 43.

The three sides a, b, and c being given, to determine the three angles A, B, *and* C. Fig. 58.

Let $a = 30°$, $b = 45°$, and $c = 22\frac{1}{2}°$.

Set off the three angles at O, which give the lines OB′, OC′, OA′, and OB″. These represent the three sides laid out on one flat surface, as shown in Fig. 55. We now wish to fold the sides a and b over, using the lines OC′ and OA′ as hinges, as it were, till the two lines OB′ and OB″ meet in the line OB‴, in precisely the same manner as we treated the paper model.

Take any point d on OB′, and draw the line de at right angles to OC′; draw the line ef perpendicular to de. With g as centre, and radius gd, draw an arc till it meets ef in f. Join fg. Then egf will give the angle C. This triangle represents a section through one-half of the trihedral angle taken perpendicular to the line OC′ "constructed" on the H.P., and if we were to imagine the line ge to represent a hinge, and rotate the triangle till the point f is immediately over the point e, it would give the position of the line OB‴; draw this line through e, which represents the plan of the edge B′ of the trihedral angle.

From e draw the line eh at right angles to OA′, and draw

the line *ck* perpendicular to *ch*. With *e* as centre, and radius *ef*, draw an arc till it meets *ck* in *k*. Join *kl*. *elk* then gives the angle A; and the triangle *elk* represents a section through the other half of the trihedral angle, taken perpendicular to the line OA′, and "constructed" on the same plane as *b*. If we were to rotate this triangle on the line *el* till it is perpendicular to the plane *b*, the point *k* would meet the point *f* in *e*.

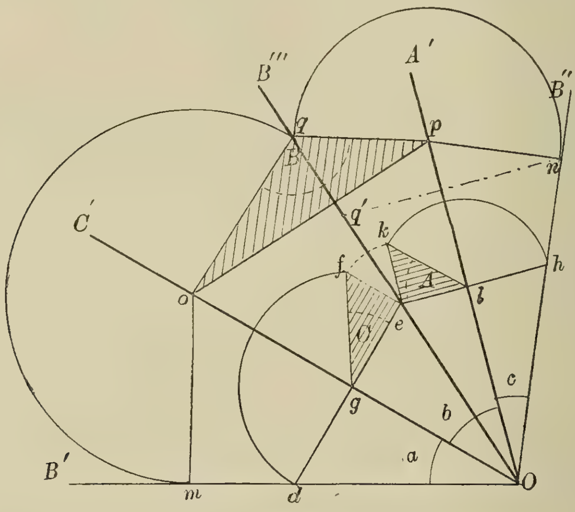


Fig. 58.

The points *d* and *h* will be found to be equidistant from O.

To obtain the angle B. Take any two points *m* and *n* equidistant from O, and at these points draw the lines *mo* and *np* perpendicular to the lines OB′ and OB″ till they meet the lines OC′ and OA′ in *o* and *p*. Join *op*. With *o* as centre, and radius *om*, draw an arc till it meets OB‴ in *q*. With *p* as centre, and radius *pn*, draw another arc till it meets OB‴ in *q*. Join *qo* and *qp*. *oqp* gives the angle B; and the triangle *opq* is a section through the trihedral angle taken in a plane perpendicular to the edge OB‴ of the trihedral angle, *i.e.* where the faces *a* and *c* join each other. If this triangular section were rotated on the line *op* it would meet OB‴ in *q*′.

This section could also be obtained by first drawing the line op at right angles to the projection of OB‴, and then drawing the lines om and pn perpendicular to OB′ and OB″.

Problem 44.

The two sides a and b and the included angle C *being given, to determine the angles* A *and* B *and the side c.* Fig. 59.

Let $a = 36°$, $b = 45°$, and $C = 70°$.

From O set off the angles of a and b, which give the lines

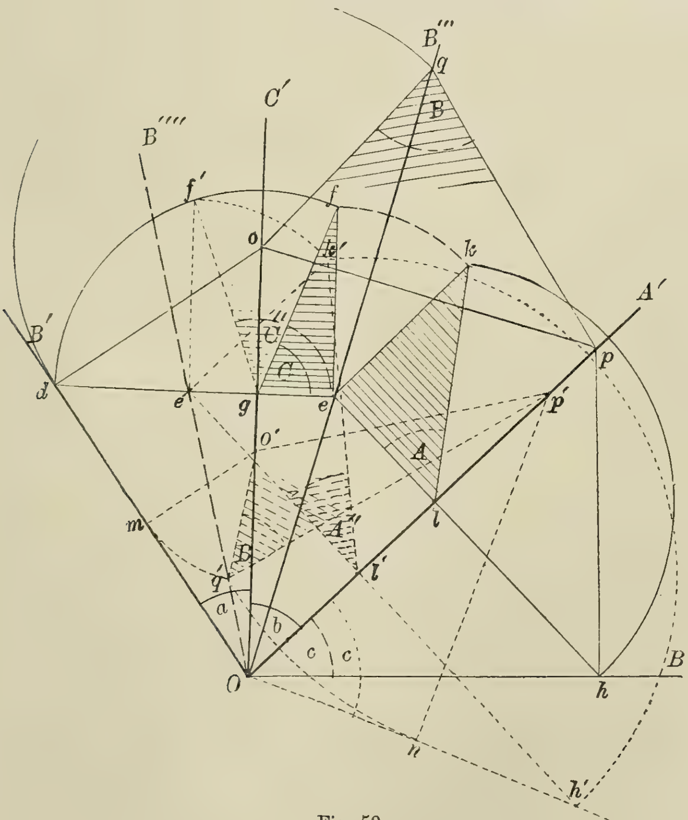


Fig. 59.

OB′, OC′, and OA′. These represent the two given sides lying

flat upon the same plane. Now rotate the side *a* on the line OC′ till it makes with *b* the given angle C. Take any point *d*, and draw *de* at right angles to OC′, and at *g* construct the given angle C by drawing the line *gf* at an angle of 70° with *ge*. With *g* as centre, and radius *gd*, draw an arc till it meets *gf* in *f*. From *f* draw *fe* perpendicular to *de*. This gives the position of the edge OB‴, as explained in the preceding problem.

Draw OB‴. From *e* draw *eh* at right angles to OA′. With *e* as centre, and radius *ef*, draw an arc till it meets *ek* drawn perpendicular to *eh* in *k*. Join *kl*. *elk* gives the angle A. With *l* as centre, and radius *lk*, draw an arc till it meets *eh* in *h*. Draw a line from O through *h*. This gives the side *c* "constructed" on the same plane as *b*.

As the points *d* and *h* are both equidistant from O, they will answer the same purpose as *m* and *n* in the preceding problem, for finding the angle B. Draw *do* and *hp* perpendicular to the lines OB′ and OB″. Join *op*. With *o* as centre, and radius *od*, draw an arc till it meets OB‴ in *q*. If we were to take *p* as centre, and radius *ph*, and draw an arc, we should find that it would also meet OB‴ in *q*. Join *qo* and *qp*. Then *oqp* gives the angle B, and the triangle *oqp* is the section through the trihedral angle perpendicular to the edge OB‴, as explained in the preceding problem.

Problem 45.

To solve the preceding problem when the angle C *exceeds* 90°.

Let C = 110°. Fig 59.

This proposition is given because students sometimes find a difficulty in solving the problem under these conditions.

Note.—The alternative construction is shown by dotted lines.

Having drawn the sides *a* and *b*, the line *de*, and the arc *df*, as before, proceed as follows. At *g* draw a line *gf′* making with *ge* the given angle (110°). Where this line meets the arc

at f' draw the line $f'e'$ perpendicular to dg and meeting it in e'. Through e' draw the line OB'''', which is the position of the edge B''', according to the altered conditions.

From e' draw the line $e'h'$ at right angles to OA', intersecting it in l'. From e' draw $e'k'$ perpendicular to $e'h'$. With e' as centre, and radius $e'f'$, draw an arc till it meets $e'k'$ in k'. Join $k'l'$. Then $e'l'k'$ is the angle A', *i.e.* A according to the new conditions. With l' as centre, and radius $l'k'$, draw an arc till it meets $e'h'$ in h'. Draw a line from O through h'. This gives the side c "constructed" on the same plane as b, according to the new conditions.

To obtain the angle B'. Take any two points m and n equidistant from O, and draw the lines mo' and np' perpendicular to Od and Oh', till they meet the lines OC' and OA' in o' and p'. Join $o'p'$. With o' as centre, and radius $o'm$, draw an arc till it meets OB'''' in q'. With p' as centre, and radius $p'n$, draw an arc till it meets OB'''' in q'. Join $o'q'$ and $p'q'$. Then $o'q'p'$ is the angle B, according to the new conditions.

It will be noticed that the point q' is found in this instance by arcs drawn in the opposite direction to those shown in Prob. 43. This is done simply to save space and confusion of lines; the result is the same whichever way the arcs are drawn. This should be tested by the student.

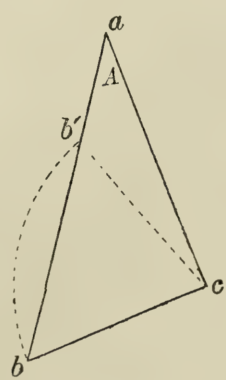


Fig. 60.

Problem 46.

To solve a spherical triangle when two sides, a and b, and the angle opposite one of them, for instance A, *is given.*

Let $a = 30°$, $b = 45°$, and $A = 35°$. Fig. 61.

If we refer to Fig. 60, which represents a plain triangle constructed from similar data, we shall see that there are two

possible solutions. Let *ac* and *cb* represent the two given sides, and A the given angle opposite one of them. It is evident that the resultant triangle could be either *abc* or *ab′c*, for the two

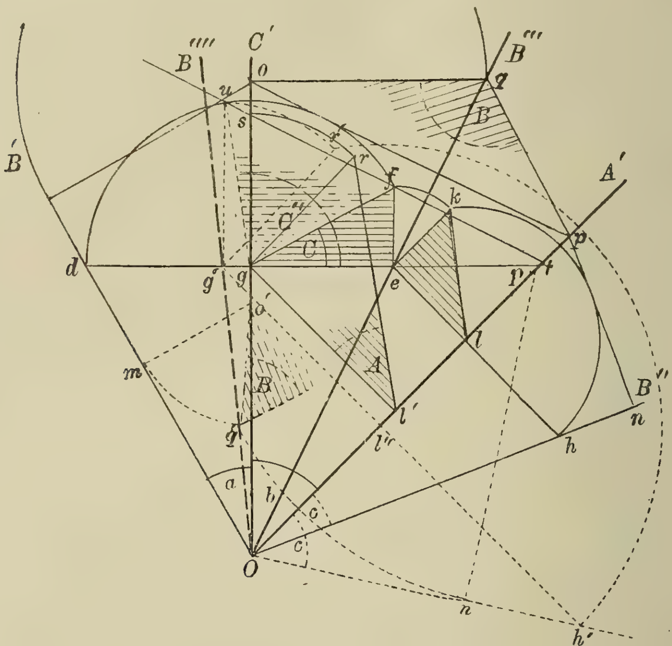


Fig. 61.

sides and given angle are the same in both. The solution of the triangle is "*ambiguous*," *i.e.* there may be two solutions.

We will give both solutions.

Set out the given sides *a* and *b* in the same plane as before, which give the positions of the lines OB′, OC′, and OA′. At any point *l′* on OA′ draw the line *l′g* perpendicular to OA′. At *g* draw the line *gr* perpendicular to *l′g*. At *l′* draw the line *l′r*, making, with *l′g*, the given angle A (35°), till it meets a perpendicular on *g* in *r*. With *g* as centre, and radius *gr*, draw an arc till it meets OC′ in *s*. Through *g* draw the line *dt* at right angles to OC′, till it meets OA′ in *t*. Draw a line from *t* through *s*. With *g* as centre, and radius *gd*, draw an arc till it meets *st* in *f*.

If a plane containing this arc, with the line *st*, be rotated on the line *dt* till it is perpendicular to the plane *b*, this arc will describe the passage of the point *d*, which, it will be seen, meets the line *st* in two successive places, viz. *u* and *f*; hence the two solutions.

From *f* draw the line *fe* perpendicular to *dt*, and join *fg*. *egf* gives the angle C. From *e* draw the line *eh* at right angles to OA′; and from *e* draw the line *ek* perpendicular to *eh*. With *e* as centre, and radius *ef*, draw an arc till it meets *ek* in *k*. Join *kl*. With *l* as centre, and radius *lk*, draw an arc till it meets *eh* in *h*. Draw a line from O through *h*. *l*O*h* gives the angle of the side *c*, and OB″ the unknown trihedral edge.

The angle B is found in precisely the same way as is shown in Prob. 43; and as the same letters are used for the corresponding points in both problems, the student will have no difficulty in completing the solution. The sections shown are also taken at the same angles to their respective edges as there shown.

Second Solution.

The construction is shown by dotted lines. Fig. 61.

We will once more imagine the arc *duf*, with the line *ut*, "rotated" on the line *dt* till it is in a plane perpendicular to the plane *b*. The point *u* is then immediately above *g*′; so, to obtain *g*′, we draw a line from *u* perpendicular to *dt*. Through *g*′ draw the line OB‴, which gives the position of the trihedral edge B, according to the second solution. Join *ug*. Then *uge* is the angle C″.

From *g*′ draw the line *g*′*h*′ at right angles to OA′, intersecting OA′ in *l*″. From *g*′ draw the line *g*′*r*′ perpendicular to *g*′*h*′. With *g*′ as centre, and radius *g*′*u*, draw an arc till it meets the line *g*′*r*′ in *r*′. With *l*″ as centre, and radius *l*″*r*′, draw an arc till it meets *g*′*h*′ in *h*′. Draw a line from O through *h*′. Then A′O*h*′ gives the side *c*, according to the second solution, and O*h*′ the unknown edge B of the trihedral angle.

To find the angle B, proceed in the same way as is shown in Prob. 44.

Problem 47.

When one side a and its two adjacent angles B *and* C *are given, to find the angle* A *and the two sides b and c.* Fig. 62.

Let $a = 35°$, $B = 45°$, and $C = 50°$.

At O set out the given side a, which gives the lines OB′ and OC′. At any points on these two lines draw the lines gm

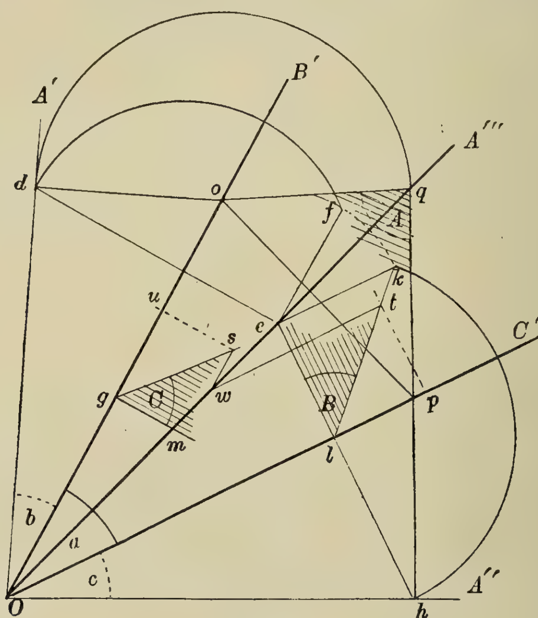


Fig. 62.

and le perpendicular to OB′ and OC′. At g draw the line gs, making, with gm, the given angle C. At l draw the line lt, making, with le, the given angle B. If these two angles be "rotated" till they are perpendicular to the side a, they form the inclination of the sides b and c with the side a, perpendicular to the edges OB′ and OC′; so, if we were to draw two

lines, one in each face, and at the same perpendicular height above the face *a*, they must meet on the unknown edge A. Set up any equal heights *gu* and *lp* on the lines OB′ and OC′, and draw lines from the points *u* and *p* parallel to *gm* and *le* till they meet the lines forming the given angles in *s* and *t*. From *s* and *t* draw lines parallel to OB′ and OC′ till they meet at *w*. This is a point on the edge A. Draw the line OA‴ through this point.

The angle A is found in precisely the same way as angle B in Prob. 43.

Problem 48.

When one side c, its opposite angle C, *and adjacent angle* A *are given, to determine the sides b and a and the angle* B. Fig. 63.

Let $c = 40°$, $A = 60°$, and $C = 50°$.

At O set out the given side *c*, which gives the lines OA′

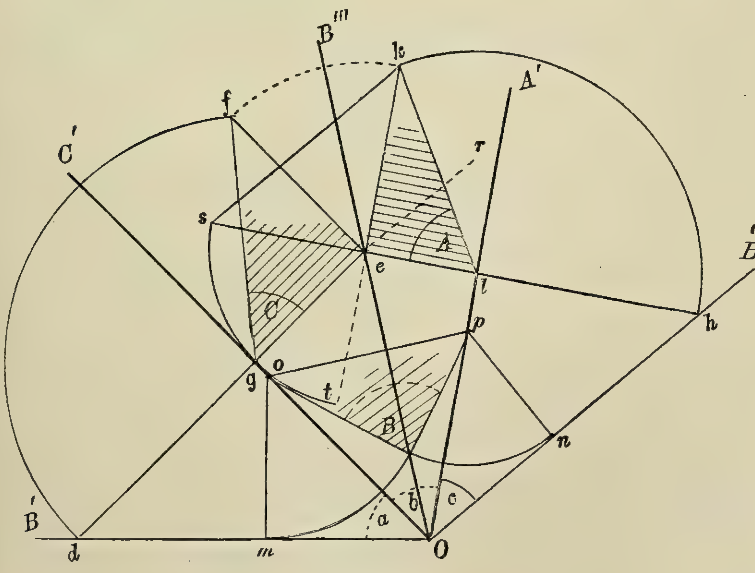


Fig. 63.

and OB″. At any point *h* draw the line *he* at right angles to

OA′, and intersecting it in *l*. At *l* draw *lk*, making, with *le*, the given angle A. With *l* as centre, and radius *lh*, draw an arc till it meets *lk* in *k*. From *k* draw *ke* perpendicular to *el*. Draw the line OB‴ through *e*.

At *e* draw *er*, making, with *el*, the given angle C. From *k* draw a line parallel to *er* till it meets *le* produced in *s*. With *e* as centre, and radius *es*, draw an arc till it meets *ke* produced in *t*. *sek* then represents the elevation, and *set* the plan of part of a cone having the same angle at its base as the given angle C. Draw a tangent OC′ to the base of this cone. This forms the trace of the side *a*, and also gives the side *b*.

From *e* draw the line *ed* at right angles to OC′ and intersecting it in *g*. From *e* draw *ef* perpendicular to *de*. With *e* as centre, and radius *ek*, draw an arc till it meets *ef* in *f*. Join *fg*. The angle *egf* is equal to the angle *ler*, *i.e.* the given angle C. With *g* as centre, and radius *gf*, draw an arc till it meets *ed* in *d*. Draw the line OB′ through *d*. This gives the side *a*.

To find the angle B, proceed in the same way as is shown in Prob. 44.

Problem 49.

The three angles A, B, *and* C *being given, to find the sides* *a*, *b*, *and* *c*.

The construction of this problem becomes very complicated if solved in direct accordance with the data as here given: it is usual to work it by its polar triangle. First obtain the supplements of the three given angles, which give the sides of the polar triangle; this can then be worked on the same principle as Prob. 43. Having obtained the three angles, their supplements give the sides of the primitive triangle.

STEREOGRAPHIC PROJECTION.

The problems in this chapter are drawn by *orthographic projection*, but there is another kind of projection called "*stereographic*," which is a species of *conical projection*; it is used for projecting on the plane of a great circle, arcs and circles that are on the surface of the sphere. This plane is called the "*plane of projection*"; one of its poles is used as a radial point, which is called the "*centre of projection*."

In Spherical Trigonometry many properties of spherical triangles become at once clear when stereographically projected, because the angle between any two circles on the surface of the sphere forms the same angle when reproduced on the plane of projection; so it is especially useful in solving the problems in Navigation and Nautical Astronomy, and belongs more to these subjects than to Solid Geometry, but we will give one illustration to show the principle.

Let AB and CD (Fig. 64) represent the planes of two

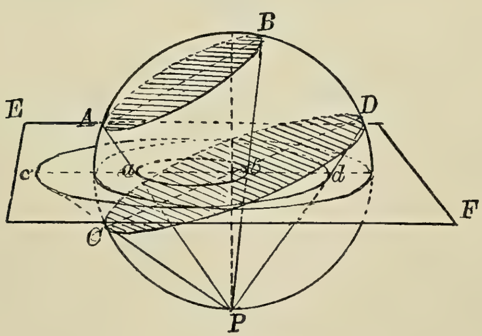


Fig. 64.

circles on the exterior of the sphere ABDPC, and EF the plane of projection passing through the centre of the sphere, of which P is the pole or centre of projection.

Then *ab* is the projection of the circle AB, and *cd* of the circle CD on the plane of projection.

HELICES.

Problem 50.

To project a helix of given pitch on a vertical cylinder. Fig. 65.

Let a'A represent the base of a cylinder, and $a'a''$ the given pitch.

From a'A draw perpendiculars, and join them by the line ae

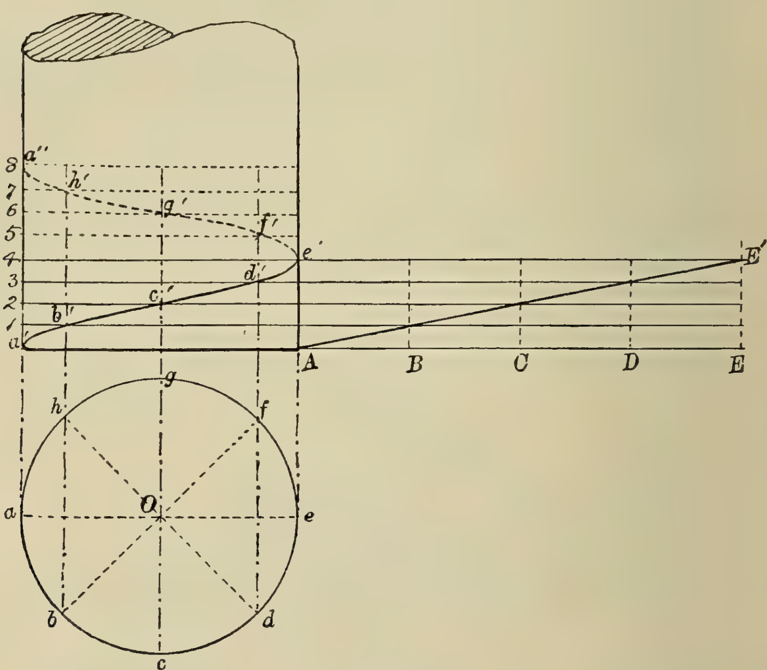


Fig. 65.

parallel to a'A. Bisect ae in O. From O as centre, and radius Oa, draw a circle, and divide it into any number of equal parts, *e.g.* eight; also divide the line $a'a''$ into the same number of equal parts, and draw lines parallel to a'A. From the various points in the plan draw lines at right angles to a'A till they intersect these parallels. These lines are called "generatrices,"

and their intersections give points in the helix, *e.g.* where the line from b meets the line from the point 1 gives the point b', etc. A fair curve drawn through these points gives the required helix.

Note.—The curve formed by the projection of a helix is called a "*sinusoid*," and the distance between a' and a'' the "pitch."

Problem 51.

To develop a given helix on a plane. Fig. 65.

Let the helix drawn in the last problem represent the given helix. To economise space we will take one-half only of the helix.

Produce the line a'A to E, and make AE equal to half the circumference of the circle (Prob. 192, Plane Geometry) forming the plan of the cylinder. At E draw EE′ perpendicular to AE, and half the height of the pitch. Join AE′. The angle EAE′ gives the angle of the pitch, and the hypotenuse AE′ one-half the true length of the helix.

If we divide AE into four equal parts, each division represents the development of a corresponding division on the surface of the cylinder, *e.g.* AB is equal to the length of the arc ab.

Problem 52.

To project the curve of a helix of given pitch on a given vertical cone. Fig. 66.

Let $aceg$ represent the plan, $a'e'$O′ the elevation, and a'8′ the pitch.

Divide the plan by radii into any number of equal parts, *e.g.* eight. Carry up projectors from these points till they meet the base in the points a', b', c', d', and e'. Join these points to the vertex O′.

Divide the pitch a'8′ into the same number of equal parts

as the plan, and draw lines from these divisions parallel to the

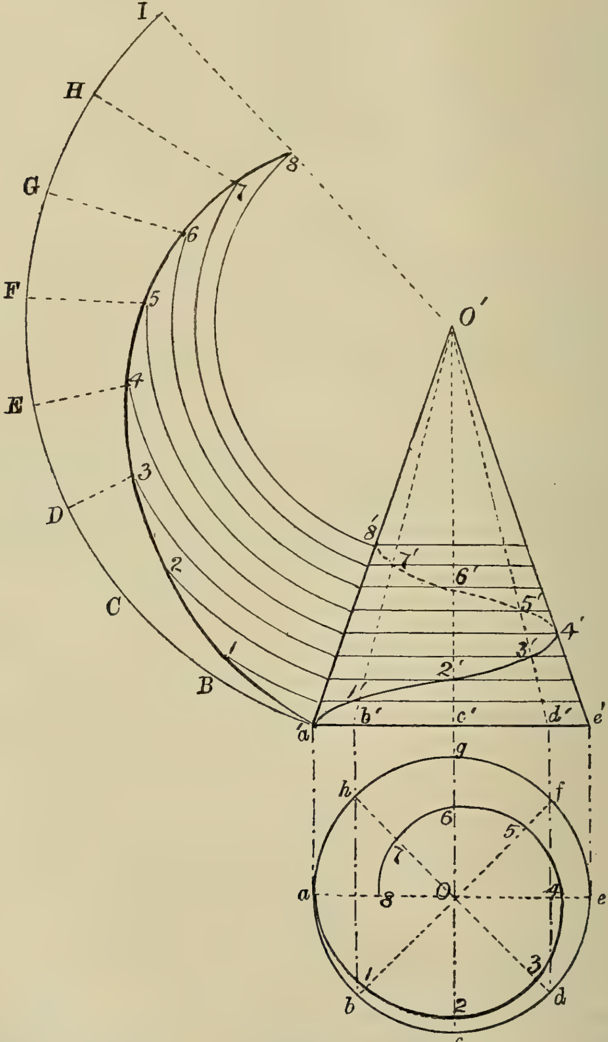


Fig. 66.

base. The intersection of these lines with those drawn to the vertex determine points in the helix; *e.g.* where the line

from b' intersects the first parallel gives $1'$, where the line from c' intersects the second parallel gives $2'$, etc. A fair curve drawn through these intersections is the required curve.

To determine the plan of this curve, fix points immediately under the intersections in the elevation on the corresponding lines in the plan; *e.g.* the point 1 on the line bO is immediately below the point $1'$ on the line b'O$'$, and the point 2 on the line cO under that of $2'$ on the line c'O$'$, etc. Join these points with a fair curve.

Problem 53.

To develop the surface of a given cone with its helix. Fig. 66.

Let Fig. 66 represent the given cone with its helix.

With O$'$ as centre, and radius O$'a'$, draw the arc a'I. Set off on this arc the length a'I equal to the circumference of the circle forming the base of the cone (Prob. 195, Plane Geometry). Join IO$'$. Then the sector a'O$'$I is the development of the surface of the cone.

Divide the arc a'I into the same number of equal parts (eight) as shown in plan, which will give the points a', B, C, etc.; and from each division draw lines converging towards the vertex O$'$. With O$'$ as centre, and each of the parallel divisions on the side of the cone as radii, draw concentric arcs till they meet in succession the lines drawn from the divisions of the arc a'I in the points 1, 2, 3, etc. A fair curve drawn through these points is the development of the helix required.

CHAPTER XXII

ON THE ALTERATION OF THE GROUND LINE

It is sometimes necessary to show more than one plan or elevation of an object. For instance, we may wish to show an end view as well as a front one, or a view of the object inclined to one or both of the co-ordinate planes: this may be accomplished by changing the position and direction of XY.

Let us again take the dressing-case we used for illustration in Chap. XVII.

In Fig. 67 A and A_1 represent the plan and elevation of

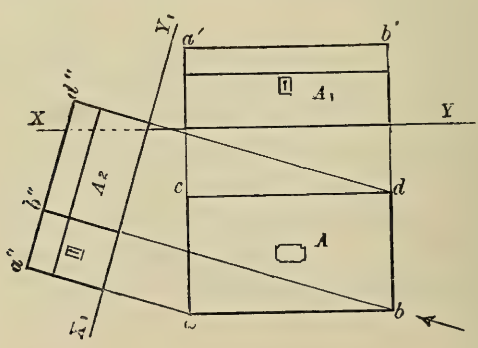


Fig. 67.

the dressing-case as shown in Fig. 2 (Chap. XVII.); we will now make another view (A_2) of the same object with its end inclined at an angle of 16° to the V.P.

Draw the line X_1Y_1 at an angle of 16° with the line *ac*, and assume it to be a new ground line. Draw the lines *aa″*, *bb″*, and

dd'' at right angles to this ground line, and make the heights of these respective lines above it equal to the corresponding lines in the elevation A_1. By joining these heights we get an elevation of the dressing-case as viewed in the direction of the arrow at *b*.

Another illustration (Fig. 68). Re-draw the elevation A_1,

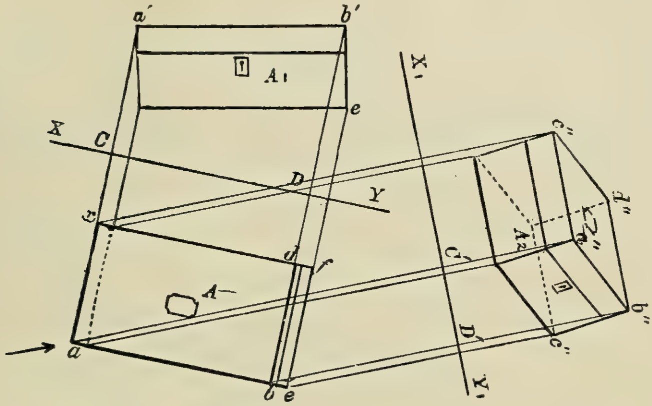


Fig. 68.

and from this construct its plan inclined to the H.P. at an angle of $12\frac{1}{2}°$.

Draw the line XY at an angle of $12\frac{1}{2}°$ with the bottom of the dressing-case. From the various points in the elevation draw lines at right angles to XY, and make the width *ca* equal to the corresponding line in the plan (Fig. 67). From these points draw lines parallel to XY. This completes the plan as required.

If the student will compare this plan and elevation with that of Fig. 4 (Chap. XVII.), he will find it is precisely similar.

From this plan and elevation project a view (A_2) inclined at an angle of 23° with the V.P.

Draw the line X_1Y_1 at an angle of 23° with the line *ef*, and assume it to be a new ground line. Draw lines from the respective points in the plan at right angles to this line, and make the heights of the points above it equal to the heights of the corresponding points in the elevation A_1; for example, make

C′a″ equal to Ca′, and D′b″ equal to Db′. By joining these points we obtain the elevation required.

We have now, from the simple elevation A_1, obtained a projection (A_2) inclined to both co-ordinate planes, *i.e.* $12\frac{1}{2}°$ with the H.P. and 23° with the V.P., by simply changing the position of the ground line.

Problem 54.

To draw an elevation of the same object when a diagonal joining the points a′ and b′ is horizontal; also a plan when the same diagonal is vertical. Fig. 69.

Note.—If we wish to incline a line in a solid and make fresh

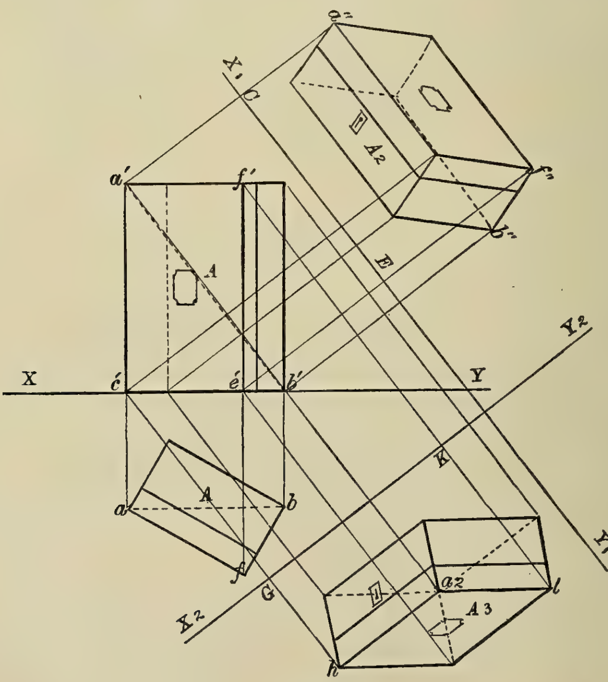


Fig. 69.

elevations and plans from it by changing the position of the

ground line, it is necessary in the first place to arrange it so that it is shown its full length ; for instance, the line $a'b'$ is the full length of the diagonal to which this problem refers, and we obtain its full length by making it parallel to the V.P., as shown by the line ab in plan.

First draw a plan of the object with a line joining the opposite points a and b parallel to the V.P., and from the various points carry up lines to form the elevation, as shown.

Join the points a' and b'. Draw the line X_1Y_1 parallel to $a'b'$ for a new ground line. Draw lines from the several points in the elevation at right angles to this ground line, and make the heights of the various points equal to the distances of the corresponding points in the plan A from the line XY ; for instance, make Ca'' equal to $c'a$ and Ef'' equal to $e'f$, etc. Join these points to obtain the elevation required.

To obtain a plan of the object with its diagonal ab in a vertical position. Draw the new ground line X_2Y_2 perpendicular to the line $a'b'$. From the various points in the elevation A_1 draw lines at right angles to this ground line, and make their distances from it equal to the distances of the corresponding points in the plan A from the line XY. For instance, make Gh equal to $c'a$, Kl equal to $e'f$, etc. By joining these points we obtain the required plan. The diagonal being vertical, will of course show as a point (a_2) on this plan.

Problem 55.

From the given projections of a solid (a hexagonal prism), to obtain other projections from them. Fig. 70.

Let $abcd$ be the plan of the solid, and $a'g'h'c'e'f'$ the end elevation. Determine a new elevation raised above the H.P., with its longer edges inclined to the V.P. at an angle of 27°.

Draw the line XY at the required angle with cd. Draw

lines from the various points in the plan at right angles to this line.

Assume *ac* to be the level of the H.P., and take the distances

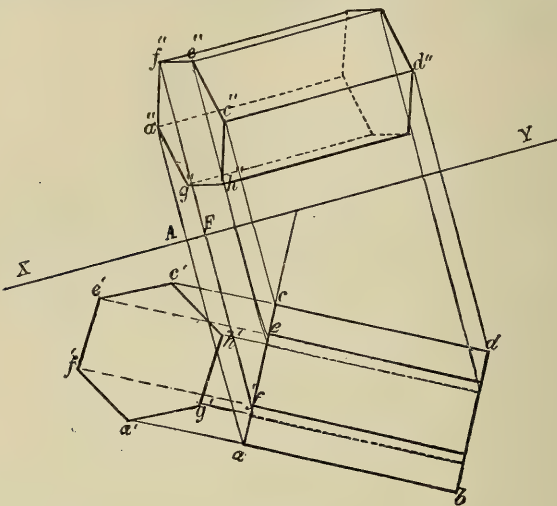


Fig. 70.

of the various points in the end elevation from *ac* for the heights of the corresponding points above XY; for instance, make Aa'' equal to aa', Ff'' equal to ff', etc. By joining these points we obtain the elevation required.

Problem 56.

From the plan and elevation of a pyramid, to draw other elevations: one, with the edge $e'b'$ parallel to both co-ordinate planes; another, with the same edge resting on the H.P., *and inclined at an angle of* 30° *with the* V.P. Fig. 71.

Let A and A_1 be the plan and elevation of the pyramid. Draw the line X_1Y_1 parallel to $e'b'$, and draw lines from the various points in the elevation A_1 at right angles to it. The height of the line $b''e''$ above X_1Y_1 will be the same as Ba', and

the height of c'' will be twice Ba'. By joining these points we obtain the first elevation required.

To project the second elevation. Draw the ground line X_2Y_2

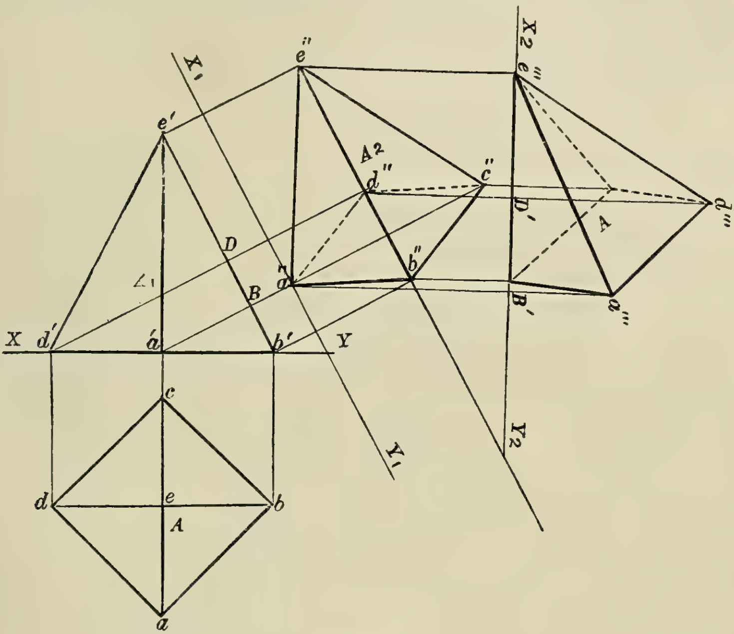


Fig. 71

at an angle of 30° with $e''b''$. Draw lines from the various points in the elevation A_2 at right angles to this line. To find the heights of the various points above the ground line, make $B'a'''$ equal to Ba' in the elevation A_1, $D'd'''$ equal to Dd', etc. By joining these points we obtain the second elevation required.

Problem 57.

From the plan and elevation of a heptagonal pyramid, to find other projections as follows:—

1. *A plan and elevation with one of the longer edges vertical, and the triangular face containing this edge parallel to the* V.P.

2. *An elevation with the same edge still vertical, but with the face containing it inclined to the* V.P. *at an angle of* 45°. Fig. 72.

Let A and A_1 be the plan and elevation of the pyramid, $a'h'$ the edge we wish to make vertical, and agh the triangular face containing it.

Draw lines from the various points in the elevation A_1

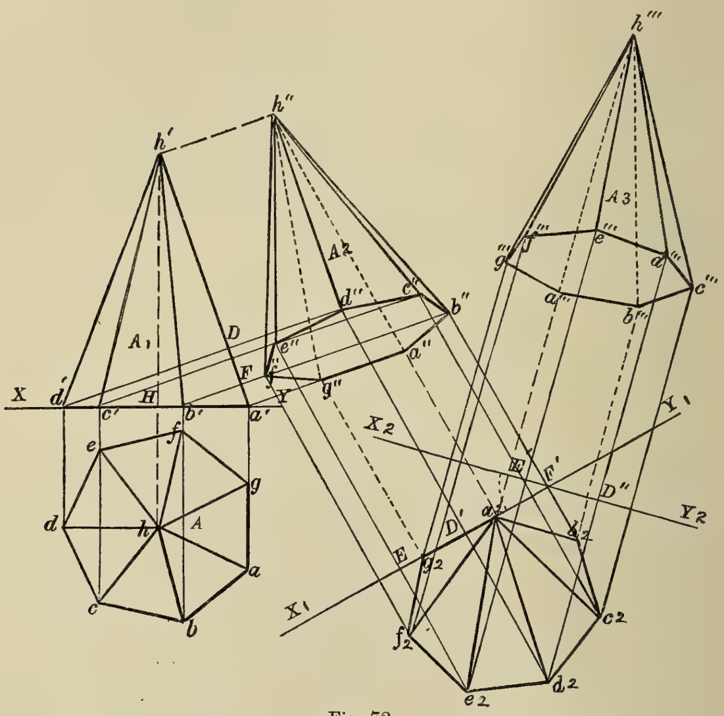


Fig. 72.

perpendicular to the line $a'h'$. Assume $a'h'$ to be the level of the H.P., and make the heights of the various points in A_2 above it equal to the distances of the corresponding points in the plan A from XY ; for instance, make $h'h''$ equal to Hh, Dd'' equal to $d'd$, etc. By joining these points we get the elevation A_2.

To project the plan. As $a''h''$ is to be the vertical edge, produce it, and draw the new ground line X_1Y_1 at right angles

to it. Draw lines from the various points in A_2 at right angles to X_1Y_1, and make their lengths below it equal to the distances of the corresponding points in elevation A_1 from the line $a'h'$; for instance, make $D'd_2$ equal to Dd', $F'b_2$ equal to Fb', etc. By joining these points we obtain the required plan.

To draw the second elevation A_3.

Draw X_2Y_2 at an angle of 45° with X_1Y_1, and draw lines from the various points in the plan at right angles to it. To find the heights of the points above X_2Y_2, make them equal to the heights of the corresponding points in elevation A_2 above X_1Y_1; for instance, make $D''d_3$ equal to $D'd''$, $E'e_3$ equal to Ee'', etc. By joining these points we obtain the required elevation.

CHAPTER XXIII

ORTHOGRAPHIC PROJECTION

THE student, before commencing this subject, should read the introduction to Solid Geometry (Chap. XVII.), in which the principles of the subject are illustrated.

In Solid Geometry objects are projected by means of parallel projectors perpendicular to two co-ordinate planes. These planes may be considered as indefinite in extent. For instance, the H.P. might be extended beyond the V.P., and the V.P. below the H.P.

To understand this fully, let us take two pieces of cardboard about 12″ square, and half-way across the middle of each cut a

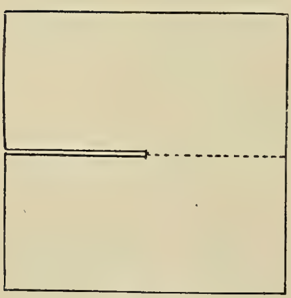


Fig. 73.

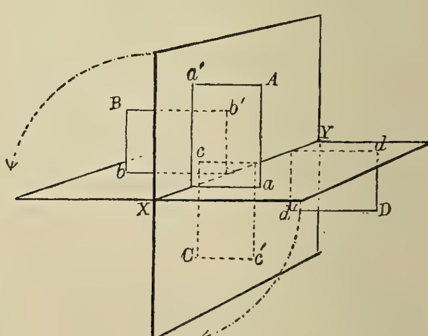


Fig. 74.

groove, as shown in Fig. 73. By fitting these two pieces together we obtain two planes intersecting each other at right angles, as shown in Fig. 74.

We have now four sets of co-ordinate planes, or, as they are called, "*dihedral angles.*"

The angle formed by the upper surface of the H.P. with the front of the V.P. is called the "*first dihedral angle.*"

The angle formed by the upper surface of the H.P. with the back of the V.P. is called the "*second dihedral angle.*"

The angle formed by the under surface of the H.P. with the back of the V.P. is called the "*third dihedral angle.*"

The angle formed by the under surface of the H.P. with the front of the V.P. is called the "*fourth dihedral angle.*"

POINTS.

We will now take a piece of cardboard $4'' \times 3''$, and place one of its shorter edges against the H.P. and a longer edge against the V.P. in the first dihedral angle, with its surface perpendicular to each plane (Fig. 74). Let the corner A represent a point we wish to project on to each plane: the top edge Aa' represents its *vertical projector*, and the point a' its *vertical projection;* the edge Aa represents its *horizontal projector*, and the point a its *horizontal projection.*

We will now place the piece of cardboard in the second dihedral angle, with a longer edge on the H.P. and a shorter one against the V.P.; its surface to be perpendicular to each plane, as before. Let its free corner now represent a point B. The point b' is its vertical projection, and the point b its horizontal projection.

Place it also in the third dihedral angle, and call its corner C. We then get c' and c for its vertical and horizontal projections.

After which, by placing it in the fourth dihedral angle, and calling the point D, we obtain its vertical and horizontal projections in d' and d.

Let us now take these two pieces of cardboard asunder. The vertical plane, with its projections, appear as shown in Fig. 75, and the horizontal plane as Fig. 76.

We will now place the vertical plane evenly on top of the horizontal plane, and prick through the vertical projections on to the horizontal plane. Remove the V.P., and letter the points

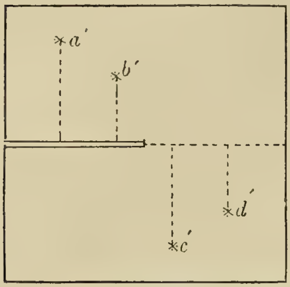


Fig. 75.

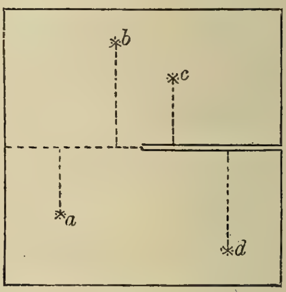


Fig. 76.

thus obtained in the same way as they are marked on the V.P., *i.e.* with a dash over each letter.

These combined projections then appear as shown in Fig. 77.

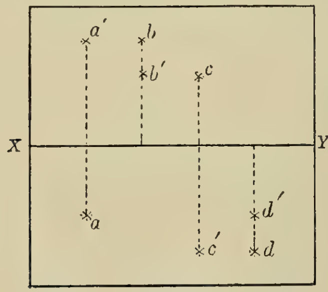


Fig. 77.

Note.—The two planes are really supposed to revolve on to each other, as shown by the arcs and arrow-heads in Fig. 74; but by the means adopted in Fig. 5 we get the same result in a more practical way.

We can see from Fig. 77 that it is possible to show the projections of all four points on one flat surface.

PROBLEM 58.

Find the projections of the four following points. Scale $\frac{1}{8}$ full size. Fig. 78.

A	to be	3″ above	H.P. and	6″ in front of	V.P.
B	,,	$7\frac{1}{2}''$	,,	6″ behind	,,
C	,,	$4\frac{1}{2}''$	,,	$7\frac{1}{2}''$ in front of	,,
D	,,	6″ below	,,	3″ ,,	,,

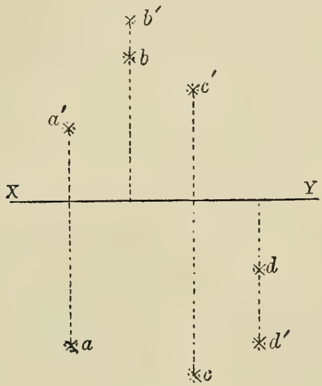


Fig. 78.

Draw XY, and the line $a'a$ at right angles to it. Measure off 3″ above XY for a', and 6″ below XY for a.

Draw a line above XY, and perpendicular to it. Measure off b' $7\frac{1}{2}''$, and b 6″ above XY.

Draw a line $c'c$ at right angles to XY; and make c $7\frac{1}{2}''$ below, and c' $4\frac{1}{2}''$ above XY.

Draw a line below XY, and perpendicular to it. Measure off d 3″ below, and d' 6″ below XY.

PROBLEM 59.

The projections of four points being given, one in each dihedral angle, to find their distances from XY. Figs. 79 and 80.

Let A be in the first dihedral angle, B in the second, C in the third, and D in the fourth.

Draw two lines at right angles to each other (Fig. 80), to

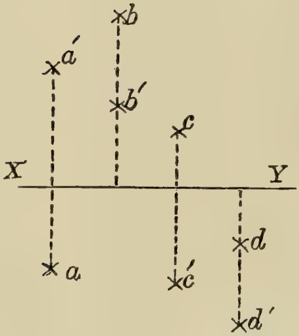


Fig. 79.

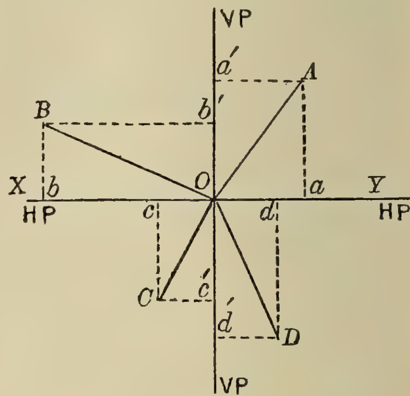


Fig. 80.

represent a side view of the H.P. and V.P. O is the position of XY.

Set off Oa from O (Fig. 80) equal to the distance a is below XY (Fig. 79). At a erect a perpendicular aA equal to the height of a' above XY (Fig. 79). Draw a'A parallel to the H.P. Join AO, which is the distance required.

Proceed in the same manner with the other three points. Then BO is the required distance of B, CO of C, and DO of D from XY.

LINES.

To illustrate the projection of lines, we will for the present restrict ourselves to the two co-ordinate planes of the first dihedral angle (Fig. 81).

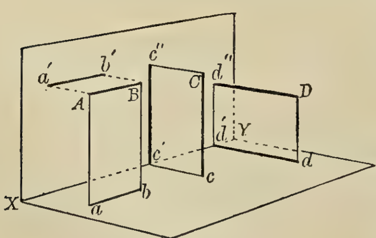


Fig. 81.

Take the piece of card-board we used for the points, and place it with one of its shorter edges on the H.P., with its surface parallel to the V.P. Let the top edge AB represent a line we wish to project. The edges Aa and Bb will then represent

the horizontal projectors, and the line ab its horizontal projection. If we draw lines Aa' and Bb' perpendicular to the V.P. from the points A and B, they represent the vertical projectors, and the line $a'b'$ its vertical projection.

We will now place the piece of cardboard touching both planes, with one of its shorter edges on the H.P., and its surface perpendicular to both planes. Let the edge Cc represent the line to be projected, Cc'' and cc' represent the vertical projectors, and the line $c'c''$ its vertical projection. The point c on the H.P. is called the "*horizontal trace*" of the line.

Note.—The point where a line, or a line produced, would meet either plane is called the "*trace*" of that line: if the point is on the H.P., it is called the "*horizontal trace*" (H.T.); and if it is on the V.P., the "*vertical trace*" (V.T.) The same thing applies to the projection of planes.

Now place the piece of cardboard with one of its longer edges on the H.P., and its surface perpendicular to both planes. Let the top edge Dd'' represent the line to be projected. The edges Dd and $d''d'$ represent the horizontal projectors, and the line dd' its horizontal projection. The point d'' is its vertical trace.

Fig. 82 represents the co-ordinate planes opened out into

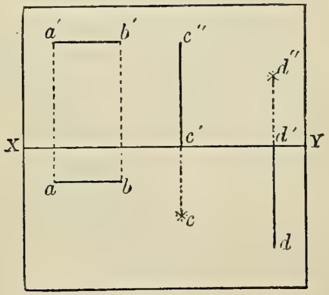


Fig. 82.

one flat surface. The projections below XY represent the plans of the lines, and those above XY the elevations.

We will now use the same piece of cardboard to illustrate the projections of lines inclined to one or both co-ordinate

planes (Fig. 83). In the first case we will incline it to both planes, with one of its shorter edges resting on the H.P. and parallel to the V.P.

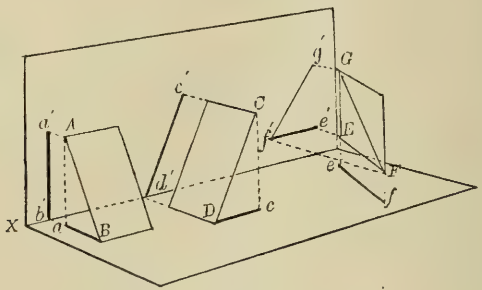


Fig. 83.

Let the edge AB represent the line to be projected. *a*B is its horizontal, and *a′b′* its vertical projections.

Then incline it to the H.P., with one of its shorter edges still on the H.P., but perpendicular to the V.P.

Let CD represent the line to be projected. The line D*c* is the horizontal, and *c′d′* its vertical projections.

Now incline it to the V.P., with its lower longer edge parallel to, but raised a little above the H.P.

Let EF represent the line to be projected. The line *ef* is its horizontal, and *f′e′* its vertical projections.

Let us now draw a diagonal GF across the piece of cardboard, and again hold it in the same position; and let GF represent the line to be projected.

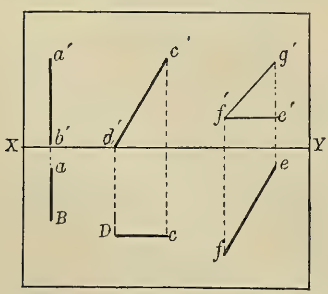


Fig. 84.

The line *ef* still represents its horizontal projection, but the line *f′g′* is its projection on the vertical plane.

Fig. 84 shows the plans and elevations of these lines, with the co-ordinate planes opened out flat.

We have now projected a line in seven distinct positions, viz. :—

Fig. 82.—	*ab* parallel to	H.P. and	parallel to	V.P.
	c perpendicular to	„	„	„
	d parallel to	„	perpendicular to	„
Fig. 84.—	*ab* inclined to	„	inclined to	„
	cd „	„	parallel to	„
	ef parallel to	„	inclined to	„
	fg inclined to	„	„	„

The student should particularly notice the difference between *AB* and *FG* in Fig. 83. Although they are both inclined to both planes, *AB* is in a plane perpendicular to the V.P., while *FG* is in one inclined to it.

We will now project these lines in the various positions to scale.

Problem 60.

To project a line AB $2\frac{1}{4}''$ *long, parallel to both the* H.P. *and* V.P., *its distances to be* $3''$ *from the* H.P. *and* $1\frac{1}{2}''$ *from the* V.P. *Scale* $\frac{1}{4}$ *full size.* Fig. 85.

Draw XY, and $1\frac{1}{2}''$ below it draw the line *ab* $2\frac{1}{4}''$ long. Draw the projectors *aa′* and *bb′* at right angles to XY, and $3''$ above it. Join *a′b′*.

Problem 61.

To project a line CD $3\frac{3}{4}''$ *long to the same scale, parallel to the* V.P. *and* $2\frac{1}{4}''$ *from it, but perpendicular to the* H.P. Fig. 85.

Fix the position of point *c* $2\frac{1}{4}''$ below XY. Draw a line perpendicular to XY, and produce same $3\frac{3}{4}''$ above it. *c* is the plan or H. trace, and *c′d′* the elevation required.

Problem 62.

To project a line EF $3''$ *long to the same scale, parallel to the* H.P. *and* $2\frac{1}{4}''$ *above it, but perpendicular to the* V.P. Fig. 85.

Below XY, and perpendicular to it, draw the line *ef′* $3''$ long.

Draw the projector $f'e'$ $2\frac{1}{4}''$ long. ef' is the plan, and e' the elevation or V. trace.

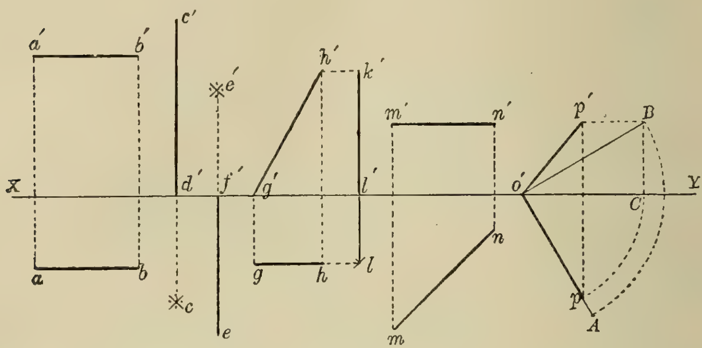


Fig. 85.

Problem 63.

To project a line GH 3″ *long to the same scale, parallel to the* V.P. *and* $1\frac{1}{2}''$ *from it, but inclined to the* H.P. *at an angle of* 60°. Fig. 85.

At any point g' on XY draw the line $g'h'$ 3″ long, and inclined to the H.P. at an angle of 60°. Let fall the projectors $g'g$ and $h'h$ at right angles to XY. Set off the points g and h $1\frac{1}{2}''$ below XY, and join them.

Problem 64.

To project a line KL 3″ *long to the same scale, inclined to the* H.P. *at an angle of* 60°, *but in a plane perpendicular to the* V.P. Fig. 85.

Draw the line $k'l$ at right angles to XY till it meets horizontal lines drawn from h' and h. $k'l'$ is the elevation, and $l'l$ the plan.

Problem 65.

To project a line MN 3″ *long to the same scale, parallel to the* H.P. *and* $1\frac{1}{2}''$ *above it, but inclined to the* V.P. *at an angle of* 45°. *The end of the line nearer the* V.P. *to be* $\frac{3}{4}''$ *from it.* Fig. 85.

Fix the point n $\frac{3}{4}''$ below XY, and draw nm 3″ long at an angle of 45° with it. Carry up the projectors perpendicular to XY, and produce them $1\frac{1}{2}''$ above it in the points m' and n'. Join $m'n'$.

Problem 66.

To project a line OP 3″ *long to the same scale, inclined to the* H.P. *at an angle of* 30°, *but in a vertical plane inclined to the* V.P. *at an angle of* 60° ; *one end of the line to be on* XY. Fig. 85.

From point o' on XY draw a line o'A 3″ long, and inclined to XY at an angle of 60°. From the same point o' draw the line o'B at an angle of 30° with XY. With o' as centre, and radius o'A, draw an arc till it meets o'B in B. Draw the line BC perpendicular to XY. With o' as centre, and radius o'C, draw an arc till it meets o'A in p. Draw the projector pp' till it meets a horizontal line drawn from B in p'. Join $p'o'$. $o'p$ is the plan, and $o'p'$ the elevation of the line required.

Problem 67.

The horizontal and vertical projections of a line being given, to find its traces. Fig. 86.

Note.—The *horizontal trace* is represented in the following examples by the letters H.T., and the *vertical trace* by the letters V.T.

Let ab represent the horizontal projection, and $a'b'$ the vertical projection.

Produce *ab* till it meets XY in E. From E let fall a line perpendicular to XY. Produce *a′b′* till it meets the perpendicular

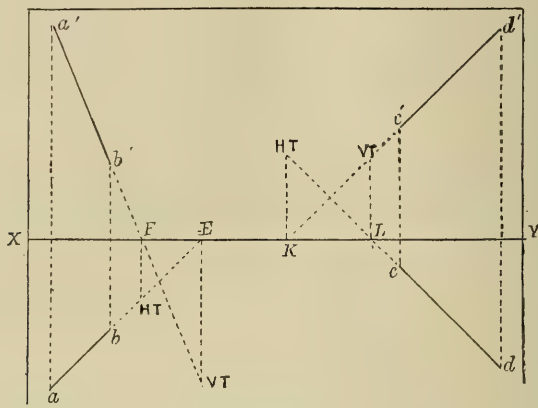


Fig. 86.

from E in V.T. (its vertical trace.) Where *a′b′* meets XY in F, let fall a perpendicular to XY till it meets *a*E in H.T. (its horizontal trace).

As *ab* is the plan, and *a′b′* the elevation of a line in the first

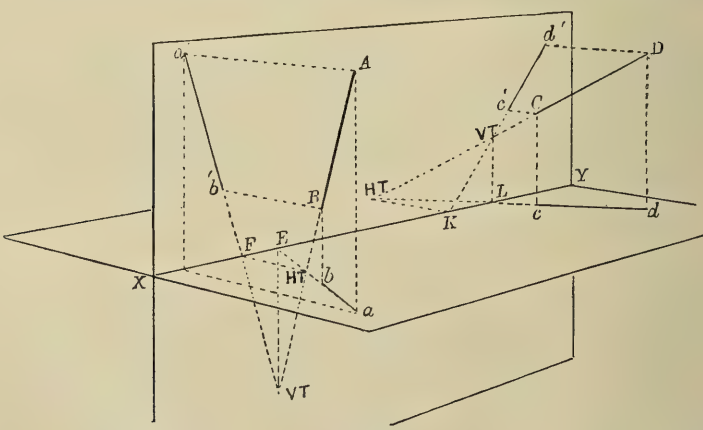


Fig. 87.

dihedral angle, the V.T. must in this instance be on the V.P. below the H.P. This will be better understood by referring to

Fig. 87, where the problem is shown in perspective, with the addition of the original line AB from which the projections are taken. The corresponding points have the same letters as are shown in the problem.

We will take another illustration of this problem. Let *cd* (Fig. 86) represent the horizontal, and *c′d′* the vertical projections of a line. Produce *c′d′* till it meets XY in K. From K erect a perpendicular to XY till it meets *cd* produced in H.T. (its horizontal trace). Where *cd* produced meets XY in L, erect a perpendicular to XY till it meets *d′*K in V.T. (its vertical trace).

As the projections of CD are in the first dihedral angle, the H.T. must be on the H.P., behind the V.P. Compare this problem also with the perspective view (Fig. 87). The student should thoroughly master the positions of these traces in the different dihedral angles, which is easily accomplished by comparing Fig. 86 with Fig. 87.

Problem 68.

To project a line $8\frac{1}{2}''$ *long parallel to the* V.P., *and* $2\frac{3}{4}''$ *from it, but inclined to the* H.P. *at an angle of* 45° ; *its lower end to be* $3''$ *above the* H.P. *Find its* H.T. *Scale* $\frac{1}{6}$ *full size.* Fig. 88.

Draw the line XY, and $3''$ above it fix the position of point *a′*. Draw *a′b′* $8\frac{1}{2}''$ long, and at an angle of 45° with XY. Let fall perpendiculars to XY from *a′* and *b′*, and produce them $2\frac{3}{4}''$ below XY. Join *ab*.

Produce *b′a′* till it meets XY in C. From C draw a line perpendicular to XY till it meets *ba* produced. This gives the H.T. This trace is on the H.P., in front of the V.P.

Problem 69.

To project a line $8\frac{1}{2}''$ *long, inclined to the* H.P. *at an angle of* 45°, *with its lower end* $1\frac{1}{2}''$ *above the* H.P. *and* $3''$ *from the* V.P.,

but in a vertibal plane inclined to the V.P. *at an angle of* 30°. *Find its* V.T. *and* H.T. *Scale* $\frac{1}{6}$ *full size.* Fig. 89.

Fix point c' $1\frac{1}{2}''$ above XY; and draw the line $c'L'$ $8\frac{1}{2}''$ long, and at an angle of 45° with XY. Let fall perpendiculars to XY from c' and L', and produce them till they meet a line cL drawn 3″ below XY and parallel to it. With c as centre, and

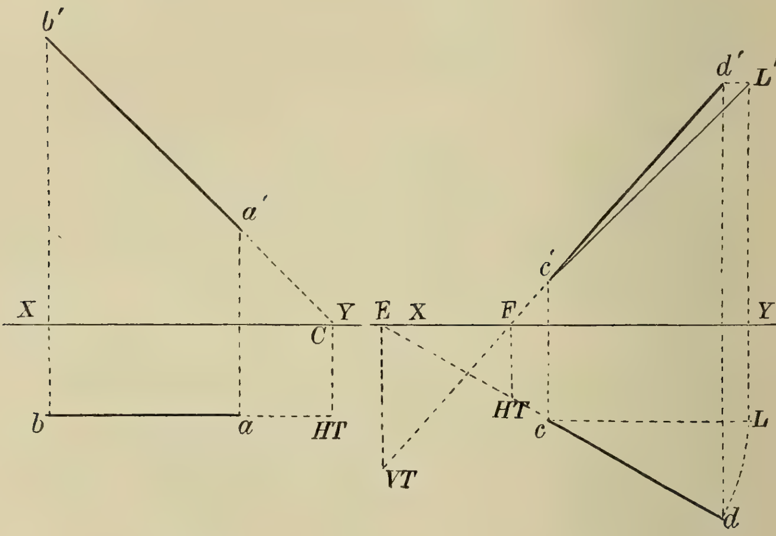


Fig. 88. Fig. 89.

radius cL, draw the arc Ld. From c draw the line cd at an angle of 30° with cL, till it meets the arc drawn from L in d. From d draw the projector dd' till it meets a horizontal line drawn from L'. Join $d'c'$.

Produce cd till it meets XY in E. From E draw a perpendicular to XY till it meets $c'd'$ produced. This gives the V.T. This trace is on the V.P., below the H.P.

Where $c'd'$ produced meets XY in F, draw a perpendicular till it meets the line produced from cd. This gives the H.T. This trace is on the H.P., in front of the V.P.

Problem 70.

To find the projections of a line, its traces being given. Fig. 90.

Let H.T. and V.T. be the traces. Let fall a perpendicular from the V.T. till it meets XY in A. Join A to the H.T. This gives the horizontal projection.

Erect a perpendicular from the H.T. till it meets XY in B. Join the V.T. to B. This is the vertical projection.

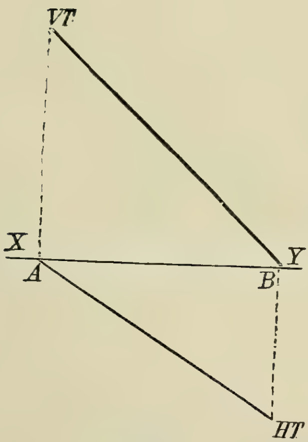


Fig. 90.

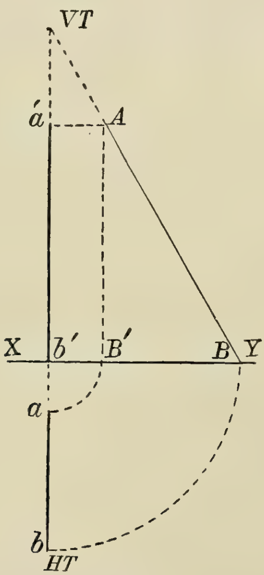


Fig. 91.

Problem 71.

The projections of a line being given, to find its traces, inclination, and true length. Fig. 91.

Let *ab* be the horizontal, and *a′b′* the vertical projections of the line. As *b′* is on XY, *b* must be the H.T. With *b′* as centre, and radii *b′a* and *b′b*, draw the arcs meeting XY in B′ and B. Draw a perpendicular at B′ till it meets a horizontal line from *a′* in A. Join AB, and produce it till it meets *b′a′*

produced. This gives the V.T. ABB′ is the inclination, and BA the true length.

Problem 72.

The projections of a line being given, to find its traces, inclination to each plane, and true length. Fig. 92.

Let ab be the horizontal projection, and $a'b'$ the vertical projection of the line.

Produce ab till it meets XY, and from this point drop a

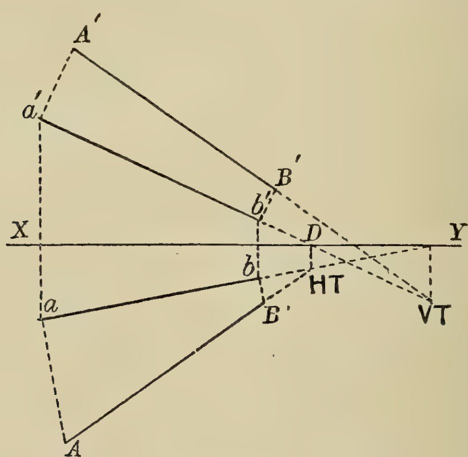


Fig. 92.

perpendicular till it meets $a'b'$ produced. This gives the V.T., which is on the V.P., below the H.P.

Where $a'b'$ produced meets XY in D, drop a perpendicular till it meets ab produced. This gives the H.T., which is on the H.P., in front of the V.P.

At a' and b' erect perpendiculars to $a'b'$. Make a'A′ equal to the distance a is from XY, and b'B′ equal to the distance of b from XY. Join A′B′. This is the inclination of the line to the V.P.

From a and b draw lines perpendicular to ab, making aA and bB equal to the distances a and b are above XY. Join AB,

the angle between AB and *ab* will be the inclination of the line to the H.P. AB is the true length of the line, and is equal to A′B′.

The Greek letter θ (theta) is generally used to express the inclination a line makes with the H.P., and the angle with the V.P. by ϕ (phi). We shall be able to remember this better if we notice that θ has a horizontal line through it, and ϕ a line in the opposite direction.

Before proceeding further with lines forming an angle with both co-ordinate planes, we will refer to Fig. 93. Let AOB

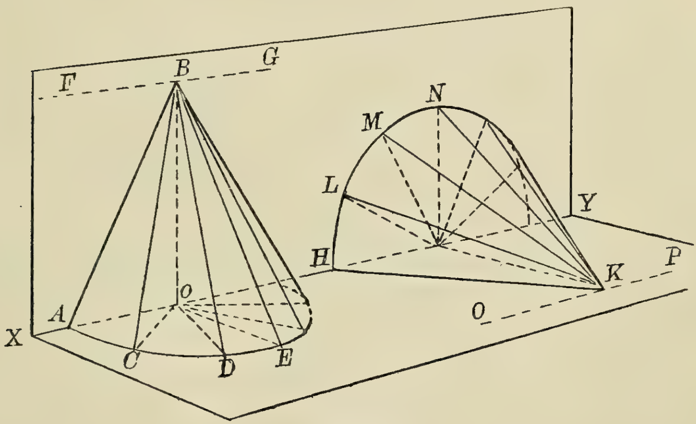


Fig. 93.

represent a 60° set-square. If we revolve it on the line OB as an axis, the hypotenuse AB will describe a semi-cone against the V.P., and the lines AB, CB, DB, etc., will all form a constant angle with the H.P., viz. 60°, although the angle they form with the V.P. will vary with each position. The height of the point B is of course constant for each position. From this we know that any line equal to AB in length, and terminated by the lines XY and a line parallel to it drawn through B, as FG, must form an angle of 60° with the H.P. This can be seen by referring to Fig. 94, which shows the plan and elevation of this semi-cone.

The same principle applies to lines forming a constant angle

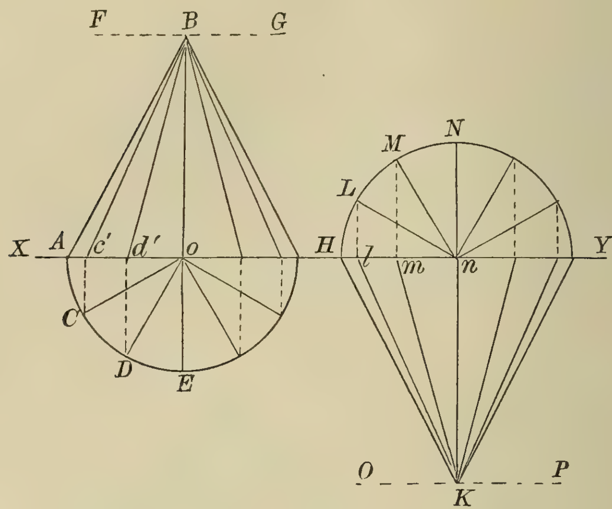


Fig. 94.

with the V.P. This is illustrated in the second part of Figs. 93 and 94.

Problem 73.

To determine the projections of a line 7″ *long, making an angle of* 50° *with the* H.P. *and* 30° *with the* V.P. *Scale* ¼ *full size.* Fig. 95.

Draw XY; and at any point A draw a line A*b*′, 7″ long, at an angle of 50° to XY. From the same point draw a line AB 7″ long, at an angle of 30° to XY. With A as centre, and radius A*b*′, draw an arc till it meets AB in B. From B draw BC perpendicular to XY. With *b*′ as centre, and radius equal to AC, draw an arc cutting XY in *a*′. Join *a*′*b*′. This is the vertical projection of the line.

From *b*′ draw *b*′*b* perpendicular to XY. With *b* as centre, and radius *b*A, draw a semicircle. This forms the plan of the

semi-cone. From a' draw $a'a$ perpendicular to XY. Join ba which gives the horizontal projection of the line.

Another Method.

Instead of setting off the angle the line forms with the V.P. below XY, it could be set off at b'. For instance, at b' draw $b'C'$ at an angle of 30° with $b'A$. Draw the line AC' perpendicular

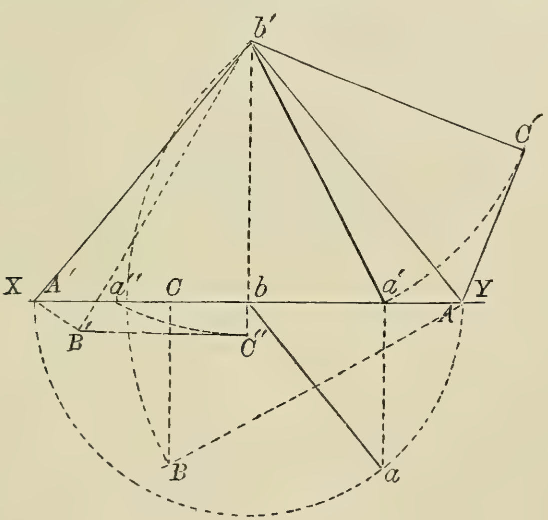


Fig. 95.

to $b'C'$. With b' as centre, and radius $b'C'$, draw an arc cutting XY in a'. Having obtained the vertical projection of the line, the plan can be obtained as shown in the first method.

Another Method.

From A' draw the line $A'b'$ at the required angle, 50°. At b' draw the line $b'B'$ at an angle of 30° with $b'b$ (the angle the line forms with the V.P.) With b' as centre, and radius $b'A'$, draw an arc till it meets $b'B'$ in B'. Draw $B'C''$ parallel to XY till it meets $b'b$ produced in C''. With b' as centre, and radius

$b'C''$, draw an arc till it meets XY in a''. This point corresponds to a', so the problem could be completed as there shown.

Problem 74.

From the horizontal projection of a line and the angle it forms with the H.P., *to find its vertical projection.* Fig. 96.

Let ab be the horizontal projection, and its angle with the H.P. 45°.

At a draw aB' at an angle of 45° with XY. With a as centre, and radius ab, draw an arc cutting XY in B. From B draw a line perpendicular to XY till it meets aB' in B′. From B′ draw a line parallel to XY till it meets a perpendicular upon b in b'. Join ab', which gives the vertical projection required.

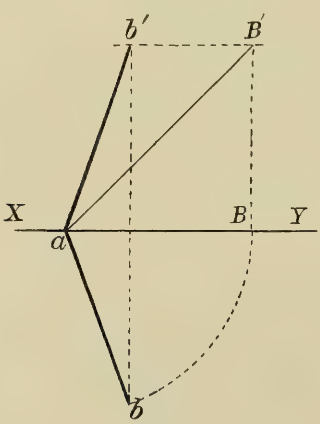


Fig. 96.

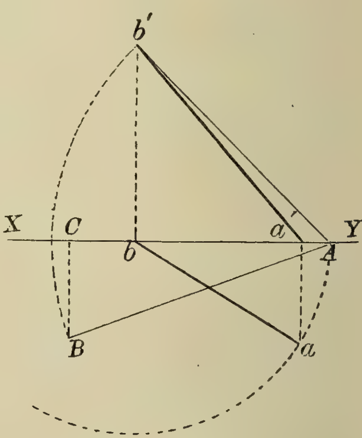


Fig. 97.

Problem 75.

From the projections of a line, to find the angles it forms with each plane, and its true length. Fig. 97.

Let ab represent the horizontal projection, and $a'b'$ the vertical projection. From b' draw the line $b'b$, and from a draw the line

aa', both perpendicular to XY. With b as centre, and radius ba, draw an arc till it meets XY in A. This represents a part of the plan of a cone. Join Ab'. This represents the inclination of the side of the same cone to the H.P. The surface of this cone is the *locus* of all lines inclined at the same angle with the H.P., *i.e.* it determines θ.

From A set off AC on XY equal to the length of $a'b'$. From C let fall a perpendicular till it meets the arc b'B drawn from centre A. Join BA. The angle BAC gives ϕ, or the inclination to the V.P. AB is the true length of the line.

Problem 76.

The vertical projection $a'b'$ of a line and θ being given, to find its horizontal projection. Fig. 98.

Draw $b'b$ perpendicular to XY. With b as centre, and the

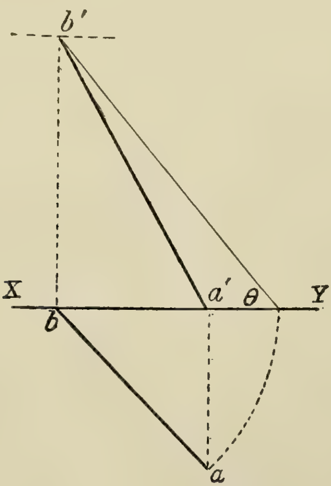


Fig. 98.

angle at θ as radius, draw an arc till it meets a perpendicular from a'. Join ba, which is the required plan.

PROBLEM 77.

The plan and elevation of an angle acb, a′c′b′ being given, to find its traces and true angle. Fig. 99.

Produce *c′a′* and *c′b′* till they meet XY in A and B; and produce *ca* and *cb* till they meet perpendiculars from A and B.

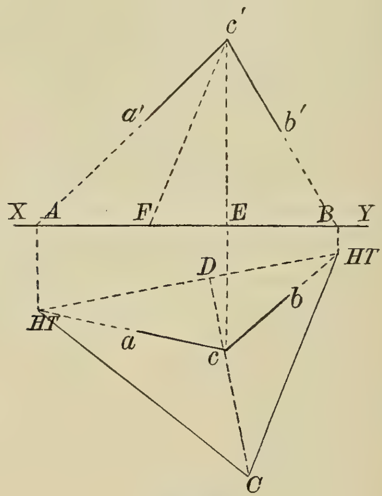


Fig. 99.

These are the traces of the lines. Join the two H.Ts. by a line, and through *c* draw the line CD perpendicular to it. Draw the projector *cc′*, and from E set off EF equal to *c*D. Join F*c′*. Set off from D a length equal to F*c′*, which determines the point C. Join C to each H.T. Then the angle at C is the true angle required.

PROBLEM 78.

The plan abc of a triangle being given,—a being 5″, b 4″, and c 2″ above the H.P.,*—to find its elevation, θ of each side, and true shape. Scale ⅛ full size.* Fig. 100.

Draw XY at any convenient distance above the plan, and draw projectors from the three points *a, b,* and *c* to the specified

heights a', b', and c' above it. Join these points, which give the elevation.

From a and b draw perpendiculars aA and bB equal in length

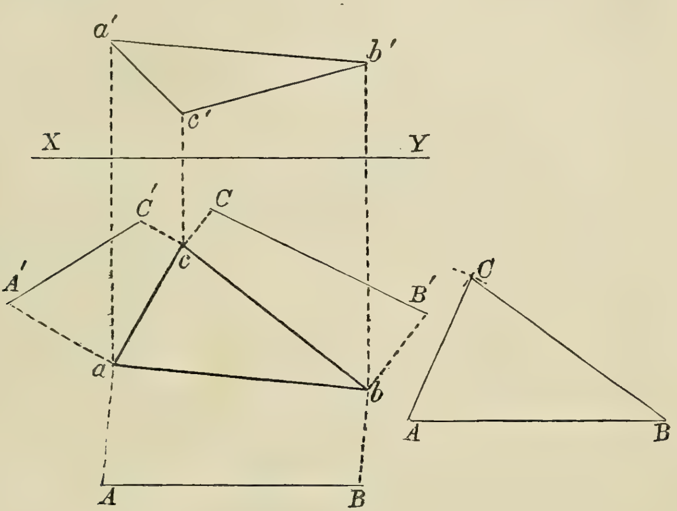


Fig. 100.

to the heights of a' and b' above XY. AB represents the true length of ab, and their inclination to each other is θ.

Proceed in the same manner with the other two sides. Having obtained the true lengths of each side, construct the triangle ABC with them. This gives the true shape of the triangle required.

Problem 79.

The projections of a line ab being given, as well as those of a point c, to project a line from this point, parallel and equal to the given line, so that when the ends of the lines are joined they will form a parallelogram. From these projections to develop the true form of the parallelogram. Fig. 101.

From c draw cd parallel to ab, and equal to it in length. Join bc and ad. This completes the plan.

Draw the projector dd'. Join $b'c'$, and from a' draw $a'd'$ parallel to $b'c'$. This completes the elevation.

Draw a diagonal bd so as to divide the plan into two equal

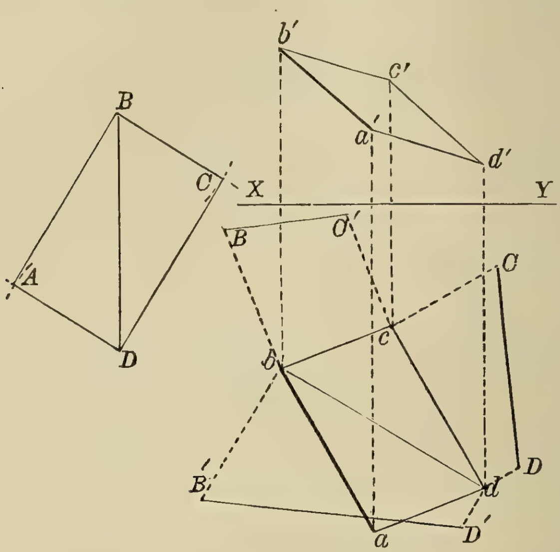


Fig. 101.

triangles. Take one of these triangles bcd, and find the true lengths of its sides, as in the preceding problem. Then construct the triangle BCD from these lengths, after which construct a similar triangle DAB on the line DB, which gives the true shape of the parallelogram.

PLANES.

The lines in which planes intersect the co-ordinate planes are called *traces:* if on the H.P., the *horizontal trace* (H.T.); and on the V.P., the *vertical trace* (V.T.) The inclination of planes is determined by means of these traces.

We will take the same piece of cardboard that we have used for our previous illustrations and place it on the H.P. and

parallel to the V.P., as A (Fig. 102). The line *ab*, where it intersects the H.P., will be its H.T.

If we place it parallel to the H.P. and perpendicular to the V.P., as B, the line *c′d′*, where it intersects the V.P., will be its V.T.

By placing it perpendicular to each plane, as C, *ef* will be its H.T. and *eg′* its V.T.

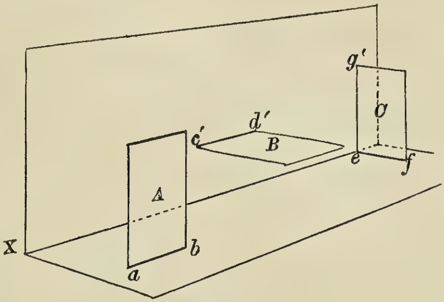


Fig. 102.

On opening these co-ordinate planes out flat these traces will appear as shown in Fig. 103.

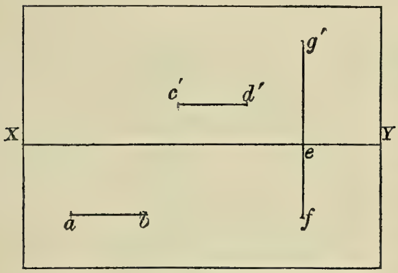


Fig. 103.

We will now place the piece of cardboard perpendicular to the H.P. and inclined to the V.P., as D (Fig. 104): *hk* will then be the H.T., and *hl′* the V.T.

Now incline it to the H.P. and make it perpendicular to the V.P.: *mn* will be its H.T. and *mo′* its V.T.

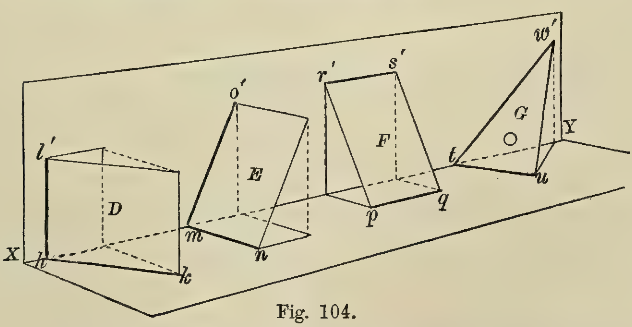


Fig. 104.

By inclining it to both planes, with its shorter edges parallel to XY, *pq* will be the H.T. and *r′s′* the V.T.

For our next illustration we will take a 60° set-square G, as a right angle will not fit closely to the two planes in this position; *tu* will be the H.T. and *tw′* the V.T.

If we now open the planes as before, these traces will be shown as in Fig. 105.

From these illustrations we can deduce the following facts—

A plane can have no trace on the plane it is parallel to (see A and B, Fig. 102).

If traces are not parallel to XY, they must intersect each other on that line (see D, E, and G, Fig. 104).

If the traces of a plane are in one straight line when the H.P. and V.P. are opened out so as to form one continuous surface, the angles the plane forms with each co-ordinate plane must be equal.

When a plane is perpendicular to either co-ordinate plane,

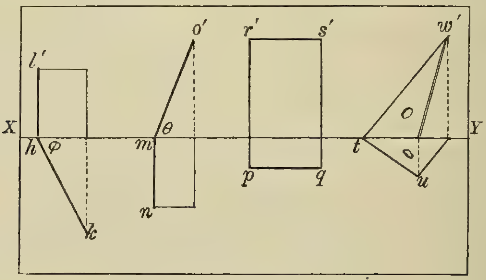


Fig. 105.

its inclined trace will always give the amount of its inclination to the other co-ordinate plane; *e.g.* *hk* (Fig. 105) forms with XY the angle ϕ or its inclination to the V.P., while *mo′* forms with XY the angle θ or its inclination to the H.P.

When a plane is inclined to both planes, but has its traces parallel to XY, the sum of its inclinations, *i.e.* $\theta + \phi = 90°$; as F (Fig. 104).

The traces of planes inclined to one or both planes are not supposed to finish at XY; they are indefinite, and are generally produced a little beyond XY.

Problem 80.

To find the traces of the following planes. Scale $\frac{1}{4}$ full size.
Fig. 106.

A, 3″ × 2¼″ perpendicular to the H.P. and inclined to the V.P. at 60°.
B, 3¾″ × 1¾″ „ „ V.P. „ „ H.P. „ 45°.
C, 4½″ × 3″ inclined „ H.P. at 60° with shorter edges parallel to each plane.

Draw XY; and at any convenient point *a* draw *ab* 3″ long,

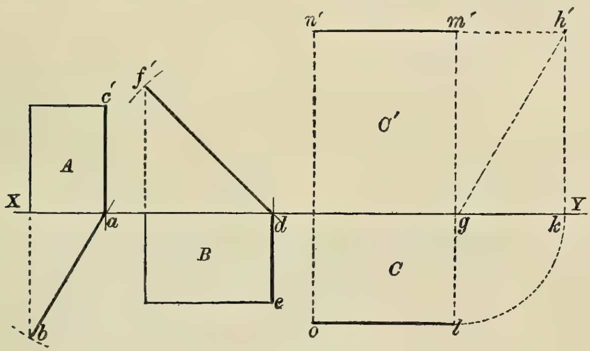


Fig. 106.

and at an angle of 60° with XY. From *a* draw *ac′* 2¼″ long. Then *ab* is the H.T., and *ac′* the V.T. of A.

From any convenient point *d* draw *df′* 3¾″ long, and at an angle of 45° with XY. From *d* draw *de* 1¾″ long. Then *de* is the H.T., and *df′* the V.T. of B.

Take any point *g* on XY, and draw *gh′* 4½″ long at an angle of 60° with it. From *h′* draw *h′k* perpendicular to XY; and from *g* as centre, with radius *gk*, draw an arc till it meets a perpendicular from *g* in *l*. From *g* draw a perpendicular till it meets a horizontal line from *h′* in *m′*. Draw *m′n′* and *lo*, each 3″ long, parallel to XY. Then *m′n′* will be the V.T., and *ol* the H.T. of C.

We will now proceed with planes that are inclined to both planes of projection: they are called *oblique planes.* Let us take a 60° set-square and place it so as to fit closely against both

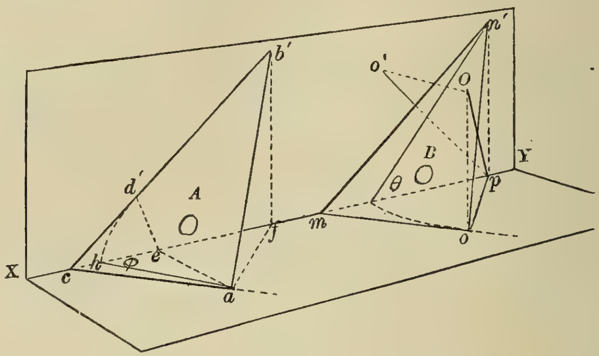


Fig. 107.

planes, as shown at A (Fig. 107). *ca* will be the H.T. and *cb′* the V.T.

The inclination of a plane to the co-ordinate plane containing its trace is the angle between two lines perpendicular to the trace, one in the co-ordinate plane and one in the plane itself.

The line *ca* (Fig. 107) is the H.T. of the plane A, and *ab′* is a line in the plane A, and *af* a line in the H.P., both perpendicular to the H.T.; therefore *b′af* is the angle A forms with the H.P.

Problem 81.

From the H.T. *and* V.T. *of a given plane, to determine the angles it forms with the two planes of projection;* i.e. *find* θ *and* ϕ. Fig 108.

Let *ca* be the H.T., and *cb′* the V.T. of the given plane.

From any point *a* in the H.T. draw *af* in the H.P. perpendicular to it. With *f* as centre, and radius *fa*, draw an arc till it meets XY in *g*. At *f* draw a perpendicular to XY till it meets the V.T. in *b′*. Join *gb′*. Then *fgb′* will give θ.

From any point d' in the V.T. draw $d'e$ in the V.P. perpendicular to it. With e as centre, and radius ed', draw an arc till it meets XY in h. From e draw ek perpendicular to XY, and join hk. Then ehk will give ϕ.

The two set-squares (Fig. 107) show this construction in perspective. A illustrates the method of obtaining ϕ, and B the way to find θ.

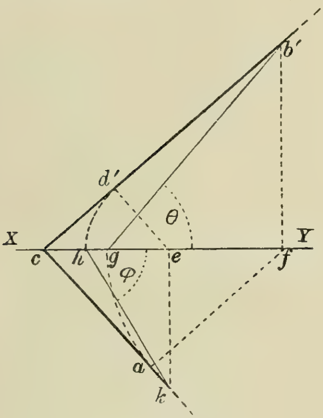


Fig. 108.

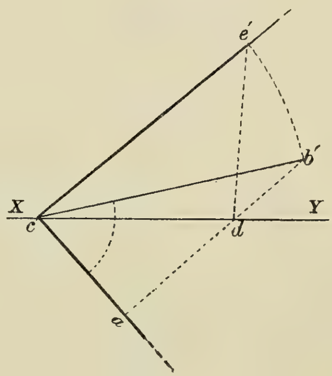


Fig. 109.

Problem 82.

To find the true angle between the H.T. *and* V.T. *of a given plane* (e.g. *the angle* $b'ca$ *in the plane* A. Fig. 107.) Fig. 109.

Let ac and ce' be the H.T. and V.T.

From any point a in the H.T. draw ad in the H.P. perpendicular to it till it meets XY in d. From this point draw a perpendicular to XY in the V.P. till it meets the V.T. in e'. With c as centre, and radius ce', draw an arc till it meets ad produced in b'. Join cb'. Then $b'ca$ is the angle required.

Problem 83.

To determine the traces of a plane inclined at an angle of 45° *to the* H.P., *and at an angle of* 35° *to the* V.P. Fig. 110.

Note.—In the Definitions (Chap. I.) a cone is described as being generated by the revolution of a right-angled triangle about one of its sides as an axis. The hypotenuse of this triangle is called a *generatrix.*

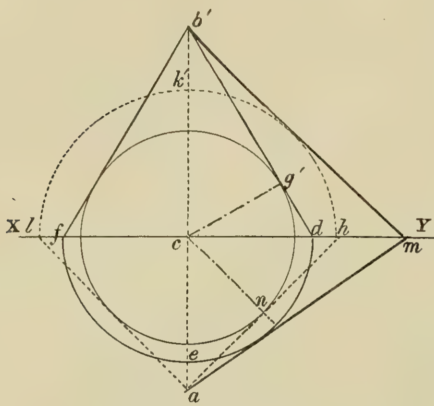


Fig. 110.

This problem is generally solved in the following manner: The generatrices of two cones forming the necessary angles to the two planes of projection are determined with their axes meeting at the same point on XY. The sides of these two cones should be tangential to a sphere, the centre of which is the point on XY in which their axes meet. The plane required is tangential to the bases of these two cones.

Draw XY. Select any point *c* for the point in which the axes of the cones meet, and draw a line through it at right angles to XY. At any point *d* on XY draw a line at an angle of 60° with it till it meets the perpendicular on *c* in *b′*. With *c* as centre, and radius *cd,* draw the semicircle *def.* Join *fb′*. Then *def* is the plan, and *fb′d* the elevation of a semi-cone.

From *c* draw the line *cg′* perpendicular to *db′*. With *c* as centre, and radius *cg′*, draw a circle. This will represent the plan and elevation of a quarter of the enveloped sphere.

Draw the line *ah,* at an angle of 45°, tangential to the plan of the sphere, cutting *b′c* produced in *a*. With *c* as centre, and

radius *ch*, draw the semicircle *hk'l*. Join *al*. Then *hal* will be the plan, and *lk'h* the elevation of another semi-cone.

From *a* draw the line *am* tangential to the semicircle *fed*; *i.e.* the base of the vertical semi-cone.

From *b'* draw the line *b'm* tangential to the semicircle *lk'h*; *i.e.* the base of the horizontal semi-cone.

Then *am* is the H.T. and *b'm* the V.T. required.

Fig. 111 is a perspective view showing this construction. The horizontal semi-cone is dotted in each instance.

There is another method of finding the traces for an oblique plane, viz. by first finding the projections of a line perpendicular to the plane required, and then drawing the traces at right angles to these projections. This will be more easily understood by referring to the set-square B (Fig. 107).

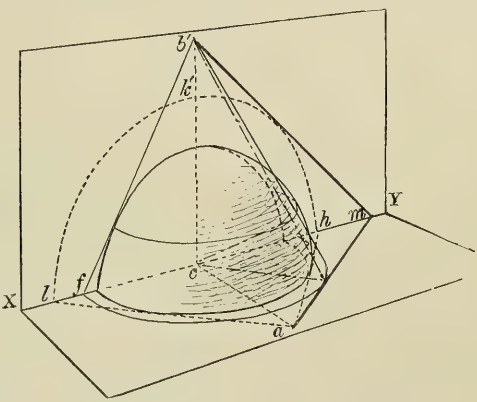


Fig. 111.

Let O*p* represent a line at right angles to *on'*, and perpendicular to the plane B. Then *op* is the horizontal projection, and *o'p* the vertical projection of this line (O*p*); and the H.T. *mo*, and the V.T. *mn'*, are at right angles to these two projections.

PROBLEM 84.

To find the traces of an oblique plane by means of the projections of a line perpendicular to it; the plane to be at 60° *with the* H.P. *and* 45° *with the* V.P. Fig. 112.

Note.—The angles in this problem are the same as those

given in the preceding one, so that a comparison can be made between them.

The angles that a line perpendicular to the plane will make with each plane of projection must first be determined.

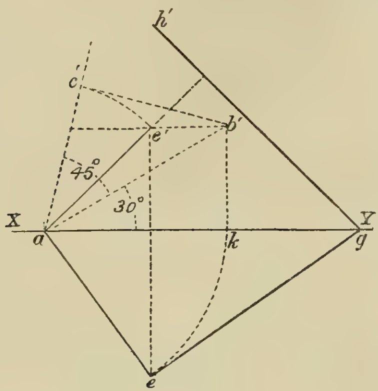


Fig. 112.

The plane is to form an angle of 60° with the H.P., so a line perpendicular to it will form an angle of 30° with the H.P. (90° – 60° = 30°). The plane also forms an angle of 45° with the V.P., so the line must also form an angle of 45° with the V.P. (90° – 45° = 45°). To find the projections of this line proceed as follows:—

Draw XY, and at any point *a* draw *ab′* at an angle of 30° with it. From the same point *a* draw *ac′* at an angle of 45° with *ab′*.

From any point *b′* in *ab′* draw *b′c′* perpendicular to *ac′*. With *a* as centre, and radius *ac′*, draw an arc till it meets a line drawn from *b′* parallel to XY in *e′*. Join *ae′*. This will be the vertical projection of the line required.

From *b′* draw a line perpendicular to XY and meeting it in *k*. With *a* as centre, and radius *ak*, draw an arc till it meets a line drawn from *e′* at right angles to XY in *e*. Join *ae*. This will be the horizontal projection of the line required.

From *e* draw the line *eg* perpendicular to *ae*. This will be the H.T. of plane required.

From *g* draw *gh′* perpendicular to *ae′* produced. This will be the V.T. of the plane required.

By comparing these traces with those shown in Fig. 110, the angles they form with XY will be found to be the same.

CHAPTER XXIV

INTERSECTION OF PLANES

If two planes intersect each other, they do so in a line common to both. If this line is parallel to the H.P., as *ab, ef,* and *gh*

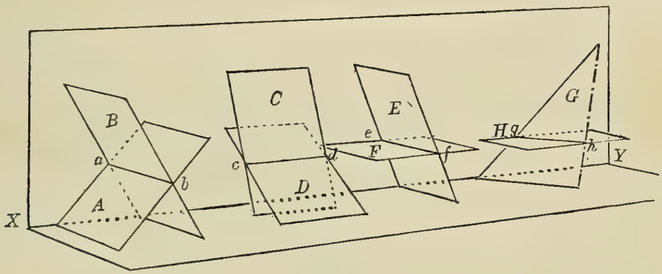


Fig. 113.

(Fig. 113), its trace will be where the V. traces of the two planes intersect; if it is parallel to the V.P., as *no* (Fig. 114), its trace

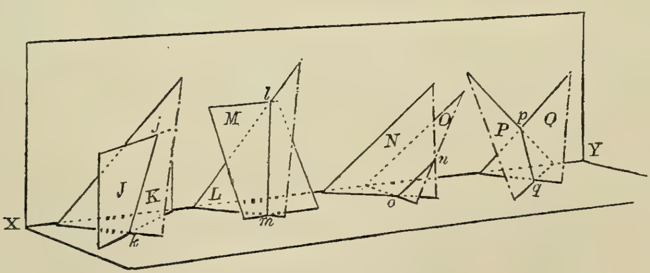


Fig. 114.

will be where the H. traces of the two planes intersect; should it be inclined to both co-ordinate planes, as *lm* and *pq* (Fig. 114), its traces will be at the intersection of both the H. and V.

traces of both planes; but if it is parallel to both planes of projection, as *cd* (Fig. 113), it will have no traces. We will now project the line of intersection of two planes, from given data, in each of the various positions illustrated in the perspective views (Figs. 113 and 114).

Note.—The planes in the following problems have corresponding letters to the planes in similar positions illustrated in Figs. 113 and 114.

Problem 85.

Project the intersection of two planes A *and* B, $3\frac{3}{4}'' \times 2\frac{1}{4}''$: A *to be inclined at an angle of* 45°, *and* B *at* 60° *to the* H.P.; *both planes to be perpendicular to the* V.P., *with their* H. *traces* 3″ *apart. Scale* $\frac{1}{4}$ *full size.* Fig. 115.

Draw XY, and mark the points e' and f' 3″ apart, inclined at 45° and 60° with XY. From e' draw the line A, and from f' the line B, each $3\frac{3}{4}''$ long. Draw $e'e$ and $f'f$ perpendicular to XY, each $2\frac{1}{4}''$ long. These will be the H. traces of the two planes. From b' draw $b'b$ at right angles to XY. *ab* is the plan of the intersection required.

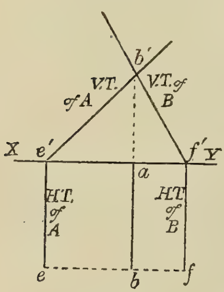


Fig. 115.

Note.—If the intersection is parallel to the H.P., its projection is also parallel to the H.T. of the intersecting plane, as *ab*, *cd*, *ef*, and *gh* (Fig. 113).

Problem 86.

Project the intersection of two planes C *and* D, $3\frac{3}{4}'' \times 2\frac{1}{4}''$: C *to be inclined at an angle of* 60°, *and* D *at* 30° *with the* H.P.; *their horizontal edges to be parallel to* XY. *Scale* $\frac{1}{4}$ *full size.* Fig. 116.

At any convenient point *g* on XY draw *gk* at an angle of 60°, and the line *gh* at an angle of 30° with XY, both $3\frac{3}{4}''$ long.

Through h draw the line lm at right angles to XY; and from k draw kl perpendicular to lm. From l draw ln parallel to kg, intersecting gh in o. Draw op perpendicular to XY. With u as centre, and distances to p, n, and g as radii, draw arcs meeting lm in the points q, r, and m. In any convenient position on XY fix the points s and t $2\frac{1}{4}''$ apart, and draw lines through them at right angles to XY. From r draw cd, and from o, $c'd'$, each parallel to XY. cd is the horizontal, and $c'd'$ the vertical projections of the intersection required. The V. traces of the two planes are found by drawing lines from the points l and h, and the H. traces by drawing lines from the points r and m, all parallel to XY.

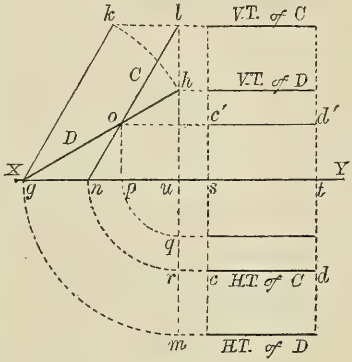


Fig. 116.

Problem 87.

Project the intersection of two planes E *and* F, $3\frac{3}{4}'' \times 2\frac{1}{4}''$: E *to be inclined to the* H.P. *at an angle of* 60°, *and perpendicular to the* V.P.; F *to be parallel to the* H.P., *and* 1″ *above it, but perpendicular to the* V.P. *Scale* $\frac{1}{4}$ *full size.* Fig. 117.

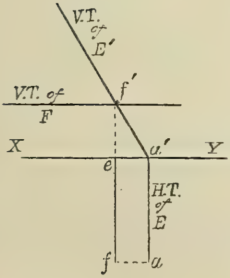


Fig. 117.

Draw XY, and at any point a' in it draw the line E′, $3\frac{3}{4}''$ long, at an angle of 60° with it. Parallel to XY, and 1″ above it, draw the line F, $3\frac{3}{4}''$ long, intersecting the line E in the point f'. These two lines are the V. traces of the two planes.

From a' draw $a'a$, $2\frac{1}{4}''$ long, perpendicular to XY: this is the H.T. of plane E. Immediately below f' draw ef, $2\frac{1}{4}''$ long, and parallel to $a'a$: this is the projection of the intersection required.

Problem 88.

Project the line of intersection of two planes G *and* H: G *to be an indefinite plane inclined at an angle of* 65° *to the* H.P. *and* 35° *to the* V.P.; H *to be* $3\frac{3}{4}'' \times 2\frac{1}{4}''$, *perpendicular to the* V.P., *parallel to the* H.P., *and* 1″ *above it. Scale* $\frac{1}{4}$ *full size.* Fig. 118.

Find the V.T., G′, and the H.T., G, of the plane G (Prob. 83 or 84). Draw the line $n'k'$ $3\frac{3}{4}''$ long, 1″ above XY, and parallel to it. This is the V.T. of plane H. Let fall projectors from n' and k' at right angles to XY, and produced $2\frac{1}{4}''$ below it, to the points n and k. Join nk. Where the line $n'k'$ intersects the line G′ in g', draw the perpendicular $g'g$. Draw the line gh parallel to the line G: this will be the horizontal projection of the intersection required. Draw the projector hh' at right angles to XY. Then $g'h'$ is the vertical projection of the intersection required.

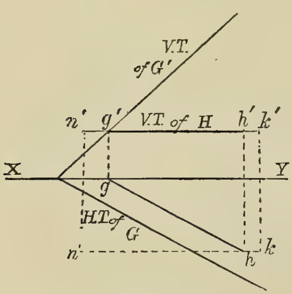


Fig. 118.

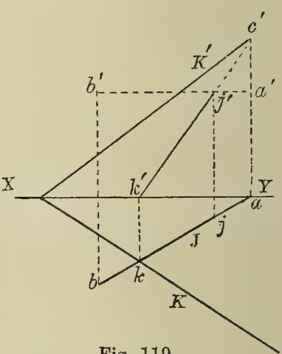


Fig 119.

Problem 89.

Project the intersection between two planes J *and* K: J *to be* $3\frac{3}{4}'' \times 2\frac{1}{4}''$, *inclined at an angle of* 30° *to the* V.P., *and perpendicular to the* H.P.; K *to be an indefinite plane, inclined at an angle of* 55° *to the* H.P., *and* 45° *to the* V.P. *Scale* $\frac{1}{4}$ *full size.* Fig. 119.

Draw XY. Find the V.T., K′, and the H.T., K, of the plane K

(Prob. 83 or 84). At any convenient point a on XY draw the line ab, $3\frac{3}{4}''$ long, at an angle of 30° with it. Draw the projectors bb' and aa' at right angles to XY, and $2\frac{1}{4}''$ in length. Join $b'a'$.

Where ab intersects the line K in k, draw the projector kk'. Produce aa' till it meets the line K′ in c'. Join $k'c'$ intersecting $b'a'$ in j'. Draw a projector from j' till it meets ba in j. Then kj is the horizontal projection, and $k'j'$ the vertical projection of the intersection required.

PROBLEM 90.

Project the intersection between two planes L *and* M : L *to be an indefinite plane inclined at an angle of* 55° *to the* H.P. *and* 45° *to the* V.P. ; M *to be* $3\frac{3}{4}'' \times 2\frac{1}{4}''$, *inclined at an angle of* 60° *to the* H.P., *with its shorter edges parallel to* XY. *Scale* $\frac{1}{4}$ *full size.* Fig. 120.

Find the V. and H. traces of the plane L (Prob. 83 or 84). At any convenient point d on XY draw the line de', $3\frac{3}{4}''$ long, at an angle of 60° with it. From e' draw the line $e'f$, perpendicular to XY. With f as centre, and radius fd, draw an arc till it meets $e'f$ produced in e. In any convenient position draw the line gh $2\frac{1}{4}''$ long, parallel to XY, and the same distance from it as point e. Draw the projectors gg' and hh' till they meet a horizontal line drawn at the height of point e'. Then gh is the H.T., and $g'h'$ the V.T. of the plane M.

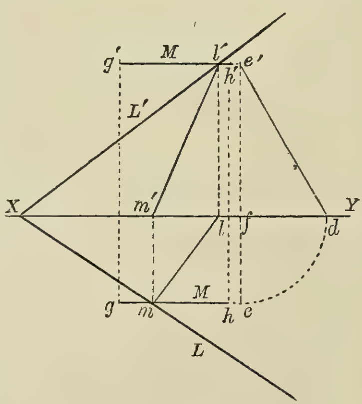


Fig. 120.

At the point m, where gh intersects the line L, draw the projector mm' ; and at the point l', where the line $g'h'$ intersects the line L′, draw the projector $l'l$. Join $m'l'$, which will be the V. projection, and ml, which will be the H. projection of the intersection required.

Problem 91.

Project the intersection of two planes N *and* O, *both inclined to the right:* N *to be an indefinite plane, inclined at an angle of* 71° *with the* H.P. *and* 29° *with the* V.P. ; O *to be also indefinite and inclined at an angle of* 60° *with the* H.P. *and* 50° *with the* V.P. ; *the points of intersection of the traces on* XY *to be* 1½″ *apart. Scale* ¼ *full size.* Fig. 121.

Draw XY, and fix the positions of the points *a* and *b* upon it, 1½″ apart. From these points draw the H. and V. traces of the two planes (Prob. 83 or 84).

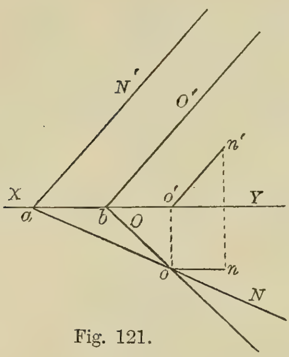


Fig. 121.

At the point *o*, where the H. traces of the two planes intersect, draw the projector *oo′*.

The V. traces of these two planes are found to be parallel to each other, consequently the line caused by their intersection must be parallel to the V. traces ; it will therefore be parallel to the V.P.

From *o* draw *on*, any length, parallel to XY ; and from *o′* draw *o′n′* parallel to the V. traces till it meets a projector from *n* in *n′*. Then *on* is the H., and *o′n′* the V. projections of the intersection required.

Problem 92.

Project the intersection of two indefinite planes P *and* Q : P *to be inclined to the left at an angle of* 58° *to the* H.P. *and* 42° *to the* V.P. ; Q *to be inclined to the right at an angle of* 63° *to the* H.P. *and* 37° *to the* V.P. ; *the intersection of their traces on* XY *to be* 3¾″ *apart. Scale* ¼ *full size.* Fig. 122.

Fix the positions of the points *c* and *d* on XY, 3¾″ apart. From *c* draw the H. and V. traces of Q, and from *d* the H. and V. traces of P (Prob. 83 or 84).

From q, where the H. traces intersect, draw the projector qq'; and from p', where the V. traces intersect, draw the projector $p'p$. Join $p'q'$, and pq. Then pq is the horizontal, and $p'q'$ the vertical projections of the intersection required.

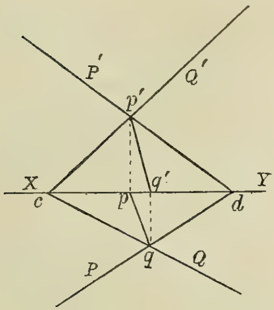


Fig. 122.

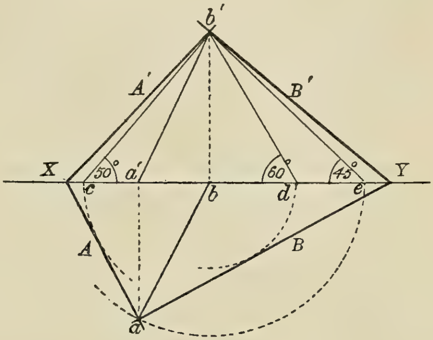


Fig. 123.

Problem 93.

To determine the traces of two planes A *and* B, *with their line of intersection:* A *to be inclined* 50°, B *at* 60°, *and the line in which they intersect* 45° *to the* H.P.; *all three inclinations to be towards the* V.P. Fig. 123.

Draw XY, and the perpendicular bb' to represent the axis of a cone. From b' the vertex draw the line cb' at an angle of 50°, db' at an angle of 60°, and eb' at an angle of 45° with XY. These three lines represent the sides of three perpendicular cones, having a common vertex at b'.

With b as centre, draw portions of the bases of these three cones. The line of intersection between the two planes is on the arc drawn from e: select any convenient point a on this arc, and join ab. Draw the projector aa', and join $a'b'$. These are the projections of the line of intersection.

From a draw a tangent to the arc drawn from c, and where it meets XY draw a line to b'. These are the traces of plane A.

From a draw a tangent to the arc drawn from d, and where it meets XY draw a line to b'. These are the traces of plane B.

Problem 94.

To determine the traces of a plane parallel to a given plane, and containing a given point. Fig. 124.

Let A and A′ represent the traces of the given plane, and aa' the projections of the given point.

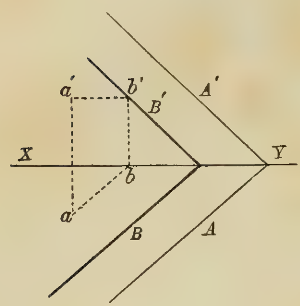


Fig. 124.

From a draw ab parallel to the trace A. From b draw a projector till it meets a horizontal line from a' in b'. Through b' draw the line B′ parallel to the trace A′, and from the point where it meets XY draw the line B parallel to trace A. B and B′ are the traces required.

CHAPTER XXV

ANGLES BETWEEN PLANES WHICH ARE NOT PARALLEL

To determine the angle between two planes intersecting each other, a line is drawn in each plane perpendicular to their line of intersection: the angle between these two lines will give the inclination of the planes to each other.

If we refer to Fig. 107, in which the line *ca* is the line of intersection between the plane A and the horizontal plane, and *ab′* and *af* are lines in each of the planes perpendicular to it, then *b′af* is the angle formed between the two planes.

Figs. 113 and 114 show the principal intersections between two planes: we will now determine the angles formed between each set of planes, in the order shown.

Fig. 115 is the projection of the planes A and B (Fig. 113). As these are both perpendicular to the V.P., their V. traces will give the angles they form with each other.

The lines *gh* and *nl* (Fig. 116) give the end view of the planes C and D (Fig. 113), consequently the angles formed by these lines will give the angles between the planes.

The planes E and F (Fig. 113) are perpendicular to the V.P., so their V. traces, as shown in Fig. 117, will give the angles between them.

Problem 95.

To determine the angle between a plane G *inclined to both planes of projection and a plane* H *parallel to the* H.P., *their traces being given.* Fig. 125.

Let G and G′ represent the traces of the plane G, and the line H′ the V.T. of plane H.

From any point a draw ab perpendicular to line G. With b as centre, and radius ba, draw the arc ac. From b draw a perpendicular to XY till it meets the line G′ in b'. Join cb', then $b'ab$ is the inclination of the plane G to the H.P. As the plane H is parallel to the H.P., the angle $e'f'g'$ is the angle between the planes required.

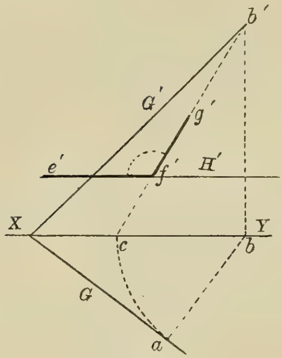


Fig. 125.

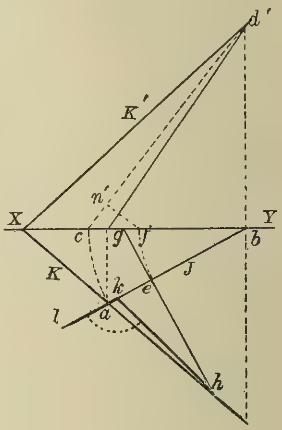


Fig. 126.

Problem 96.

To determine the angle between a plane J *inclined to the* V.P. *and perpendicular to the* H.P. *and a plane* K *inclined to both planes of projection, their traces being given.* Fig. 126.

Let K and K′ represent the traces of plane K, and J, cutting XY in b, and K in a, the H.T. of plane J.

Draw any line gh at right angles to the line J, and intersecting it in e. With b as centre, and radii ba and be, draw arcs till they meet XY in c and f. At b draw a perpendicular to XY till it meets K′ in d'. Join cd'. Then bcd' is the angle the plane K forms with the H.P. From f draw the line fn' perpendicular to cd'. Set off from e, ek equal to fn'. Join kh. Then lkh is the angle required.

The principle is this: keh is a right-angled triangle, of which

fn′ is the altitude; this triangle is rotated on the side *eh* till it coincides with the H.P., *i.e.* it is "constructed" into the H.P.

Problem 97.

To determine the angle between a plane L *inclined to both planes of projection and a plane* M *also inclined to both the* H.P. *and* V.P., *but parallel to* XY, *their traces being given.* Fig. 127.

Let L and L′ be the traces of plane L, and M and M′ the traces of plane M. Find the projections of the line of intersection between these two planes (Prob. 90), as *ab* and *a′b′*.

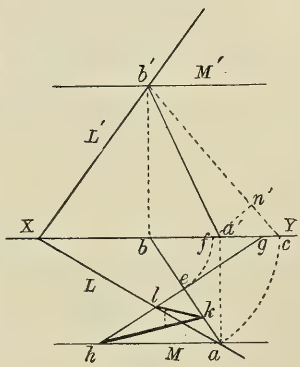


Fig. 127.

Draw the line *hg* at right angles to *ab*, and intersecting it in *e*. With *b* as centre, and radii *ba* and *be*, draw arcs till they meet XY in *c* and *f*. Join *cb′*, then *b′cb* is the inclination of plane L with the H.P. From *f* draw *fn′* perpendicular to *cb′*. From point *l*, where *hg* cuts the H.T. of plane L, set off a distance equal to *fn′*, till it meets the line *ab* in *k*. Join *lk* and *kh*. Then *lkh* is the angle required. *lkh* represents a triangle, of which *k* is the apex, rotated on the line *hl* till it coincides with the H.P.

Problem 98.

To determine the angle between two planes N *and* O, *both of which are inclined to the* H.P. *and* V.P., *with their* V. *traces parallel to each other, their traces being given.* Fig. 128.

Let N and N′, O and O′, be the traces of the two planes N and O, meeting in *a*.

At *a*, which may be any point in the H.T. of plane O, draw *ab* perpendicular to it. From *b* draw a perpendicular to XY till it

meets the V.T. of plane O in d'. Join cd', then $d'cb$ is the inclination of plane O to the H.P.

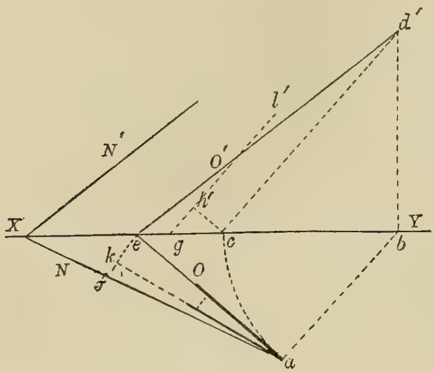


Fig. 128.

With a as centre, and radius ae, draw an arc till it meets the H.T. of plane N in f. From c set off on XY cg equal to ef, and draw gl' parallel to cd'. From c draw ch' perpendicular to gl'. From e on ef set off ek equal to ch'. Join ka. Then eak is the angle required.

Problem 99.

To determine the angle between two planes P *and* Q, *both of which are inclined to each of the co-ordinate planes, but in opposite directions, their traces being given.* Fig. 129.

Let P and P′, Q and Q′, be the traces of the two planes P and Q. Find the projections ab and $a'b'$ of the line of intersection between the two planes (Prob. 92).

Through any point e in ab draw the line hg at right angles to it. From b, with radii ba and be, draw arcs till they meet XY in the points c and f. Join cb', then $b'cb$ is the inclination of the line of intersection to the H.P.

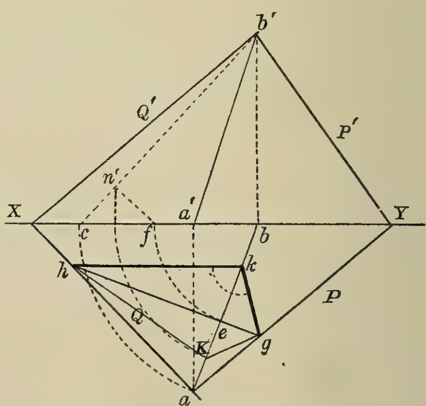


Fig. 129.

From f draw fn' perpendicular to cb'. From e along ab set off ek equal to fn'. Join kh and kg. Then hkg is the angle between the two planes P and Q required.

hkg is a triangle with an altitude equal to fn', rotated on the line hg till it coincides with the H.P. hKg is the H. projection of this triangle in its proper position.

Problem 100.

To find the traces of a plane making a given angle A *with a given plane whose traces are given.* Fig. 130.

Let H.T. and V.T. represent the traces of the given plane.

Let ab be the plan of the line of intersection between the

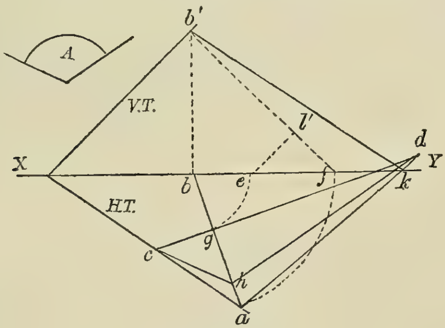


Fig. 130.

planes. At b erect a perpendicular till it meets the V.T. in b'. Draw any line cd at right angles to ab, and intersecting it in g. With b as centre, and radii bg and ba, draw arcs till they meet XY in e and f. Join fb'. At e draw the line el' perpendicular to fb'. Set off from g, towards a, gh equal to el'. Join ch. Draw hd, making with ch an angle equal to the given angle A and cutting cg in d. Join ad intersecting XY in k. Join kb'. ak is the H.T. and kb' the V.T. of the plane required.

CHAPTER XXVI

LINES INCLINED TO PLANES

The angle between a line and a plane is the angle between the line and its projection on the plane. Referring to Fig. 83, let AB represent the line and aB its horizontal projection. ABa will represent the angle between the line and the H.P.

As projectors are perpendiculars to the co-ordinate planes, the angle between a line and its longer projector is the complement of the angle between the line and its projection; *e.g.* BAa (Fig. 83) is the complement of ABa, *i.e.* BAa + ABa = 90°.

Problem 101.

To find the intersection of a given line with a given plane, the line to be horizontal and inclined to V.P. Fig. 131.

Let H.T. and V.T. represent the traces of the plane, and ab, $a'b'$, the projections of the line.

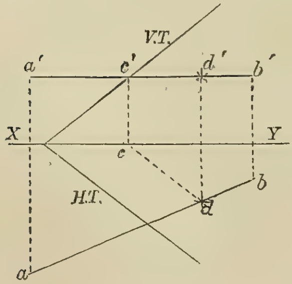


Fig. 131.

From c', where $a'b'$ cuts the V.T., draw the perpendicular $c'c$. From c draw a line parallel to the H.T. till it meets ab in d. Then d is the plan of the intersection. From d draw a projector till it meets $a'b'$ in d'. This is the elevation of the intersection required.

PROBLEM 102.

To find the intersection of a given line with a given plane, the line to be inclined to both co-ordinate planes. Fig. 132.

Let H.T. and V.T. represent the traces of the plane, and *ab*, *a′b′*, the projections of the line.

Assume a V. plane containing the given line, and find the projections of line of intersection, EC, between this assumed plane and the given plane (Prob. 89). At *d′*, where this line cuts *a′b′*, draw a projector till it meets *ab* in *d*. *d* is the plan, and *d′* the elevation of the required intersection.

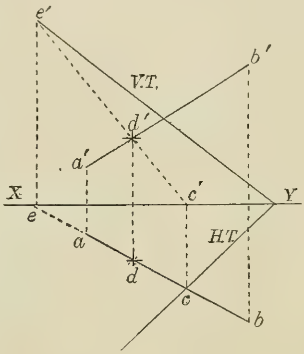


Fig. 132.

PROBLEM 103.

To find the traces of a plane perpendicular to a given line. Fig. 133.

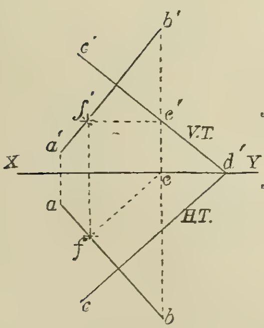


Fig. 133.

Let *ab*, *a′b′*, represent the projections of the given line.

At any point *f* in *ab* draw the line *fe* perpendicular to it. Draw projectors from *f* and *e*. Where the projector from *f* meets *a′b′* in *f′*, draw *f′e′* parallel to XY till it meets the projector from *e* in *e′*. Through *e′* draw the line *c′d′* at right angles to *a′b′*. From *d′* draw *d′c* parallel to *ef*. Then *cd′* is the H.T., and *d′c′* the V.T. of the required plane.

Problem 104.

To draw a line perpendicular to a given plane, and to determine the true length of the line. Fig. 134.

Let H.T. and V.T. represent the traces of the given plane.

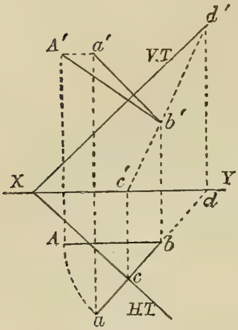


Fig. 134.

Draw ab and $a'b'$, the projections of the line, at right angles to H.T. and V.T. Produce ab till it meets XY in d. At d erect a perpendicular till it meets the V.T. in d'. Draw the projector cc', and join $c'd'$. With b as centre, and radius ba, draw an arc till it meets a line drawn from b parallel to XY in A. Draw a projector from A till it meets a line drawn from a' parallel to XY in A$'$. Join A$'b'$, which will be the true length of the line required.

Problem 105.

To determine the angle which a horizontal line inclined to the V.P. *will make with a given plane.* Fig. 135.

Let H.T. and V.T. be the traces of the given plane, and ab and $a'b'$ the projections of the line.

Draw the projections of any line AC, as ac, $a'c'$, at right angles to the traces of the plane. Through c draw the line cd perpendicular to ab. Let ad represent the axis of a cone, with a as vertex, and generatrix ac. If this line ac were rotated round the axis ad, the angle it would make with ad would be constant, and the

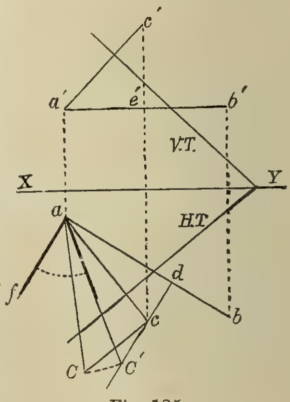


Fig. 135.

point c would always be in a plane perpendicular to the line passing through cd. If we were to rotate this line ac till it became horizontal, we should not only obtain its true length, but also the angle it forms with ab. To obtain the true length of ac, draw cC perpendicular to ac, and make it equal to $e'c'$. Join aC. This is the true length of ac. We must now transfer the point C on to the line cd. With a as centre, and radius aC, draw an arc till it meets cd produced in C′. Join aC′. The line aC′ represents the line ac in a horizontal position, and the angle C′ad is the angle the line ab forms with a perpendicular (AC) to the given plane. It has already been stated that "the angle a line will form with a plane is the complement between that line and a perpendicular to the plane"—to obtain this complement, draw af perpendicular to ab. Then faC′ is the angle required.

Problem 106.

To determine the angle which a line inclined to both planes of projection will make with a given plane. Fig. 136.

Let H.T. and V.T. represent the traces of the given plane, and ab, $a'b'$, the projections of the line AB.

Draw the lines de and $d''e'$ perpendicular to the traces: these represent the projections of a line DE perpendicular to the given plane. Find the H. traces of the lines AB and DE (Prob. 67), as c and f. Draw the line cg through f, and the line dg perpendicular to it. Let the line cg represent

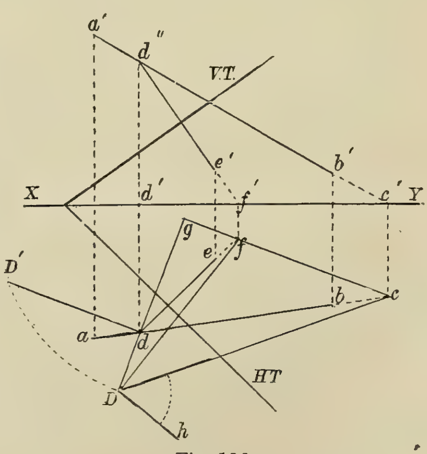


Fig. 136.

the axis of a cone, with c as vertex, and let cd be the H. projection of a generatrix of the cone. We wish to rotate this line cD till it is "constructed" on to the H.P. To do this, draw the line dD′ perpendicular to dg, and equal in length to $d'd''$ (the height of the point D above the H.P.) With g as centre, and radius gD′, draw an arc till it meets gd produced in D. Join Df and Dc, which are on the H.P., and represent the true lengths of these lines. The angle cDf is the angle formed with AB by a line perpendicular to the given plane. Draw Dh perpendicular to Df. Then cDh is the complement of the angle cDf, *i.e.* the angle required.

Problem 107.

To determine the projections of two lines from two given points outside a given plane making equal angles with the given plane, and meeting in the same point on it. Figs. 137 and 138.

Before proceeding with the construction of this problem, we will take an illustration. Let BC (Fig. 137) represent the top edge of a mirror in a vertical plane, A a point to be reflected, and E the position of the eye. The reflection of the point A in the mirror will appear to come from A′ (A′ being the same distance from the mirror as A, in a line perpendicular to its surface), whereas it really comes from point a on the mirror. If we join A and E to a, they will make equal angles with the surface of the mirror, because the angle of incidence is equal to the angle of reflection.

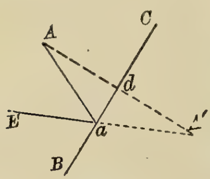


Fig. 137.

Let H.T. and V.T. (Fig. 138) represent the traces of the given plane, and a, b; a', b'; the projections of the given points.

From b draw a line cutting the H.T. at right angles in d, and produced till it meets XY in c. From d draw the projector dd', and from c the projector cc'. Join $d'c'$.

From b' draw a line at right angles to the V.T., intersecting $d'c'$ in e'. Produce this line to f', making $e'f'$ equal to $b'e'$. This is equivalent to the line AA′ in Fig. 137. The point e' on the surface of the plane represents the point d on the surface of the mirror. From f' draw a projector till it meets bc in f. Join af, and produce it till it meets XY in k, cutting the H.T. in g. Draw the projectors gg' and kk'. Join $g'k'$. Draw a line from a' to f' intersecting $g'k'$ in h'. Draw a projector from h' till it meets ak in h. The points h and h' are the projections of the point on the plane where the lines meet, answering to point a in Fig. 137.

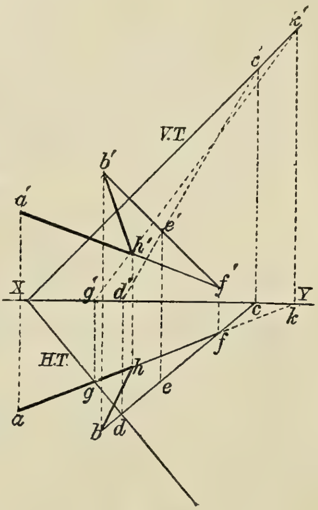


Fig. 138.

Join ah, and bh, which are the horizontal projections; $a'h'$, and $b'h'$, the vertical projections of the two lines required.

CHAPTER XXVII

LINES CONTAINED BY PLANES AND MAKING GIVEN ANGLES WITH THE CO-ORDINATE PLANES

THE principles on which solutions of this combination of figures are based will be more easily understood by reference to the perspective view (Fig. 139).

A, B, and C represent planes inclined to both co-ordinate planes. It is required to place in these planes lines inclined at a given angle to the H.P.

A point *a* in the V.T. of the plane A can be taken as one

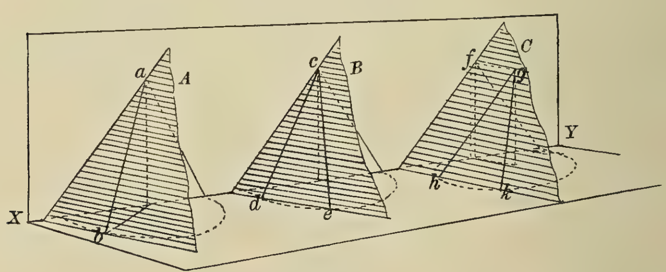


Fig. 139.

point in this line. If now a perpendicular be let fall from *a* till it meets the H.P., and a right-angled triangle with an angle θ at its base sweep out a semi-cone round this perpendicular line as an axis, the base of this semi-cone will intersect the plane A, provided the angle θ be not greater than the angle which the plane itself makes with the H.P., and where it intersects the plane, the side of the triangle *ab*, *i.e.* the generatrix of the semi-cone, will be the line required. In this case the base of the

semi-cone touches the plane in one point only, b, and the line ab is the line in the plane A inclined to the H.P. at the angle θ.

In plane B (Fig. 139) c is the vertex of the semi-cone, the base of which cuts the H.T. of the plane in two places, d and e. By joining each of these points with c we obtain two lines, both of which fulfil the required conditions, *i.e.* they both form the required angle with the H.P. and are contained by the plane B.

In plane C (Fig. 139) the vertex g is not on the V.P., as is the case with A and B, consequently we get more than a semi-cone. The base intersects the H.T. of the plane in two points h and k. By joining these two points with the vertex g we also obtain two lines, both of which are contained by the plane, and form an equal angle with the H.P.

Note.—The angle the required line forms with the H.P. must not exceed that formed by the plane containing it with the H.P., but it may be at the same angle as is the case with the line ab and plane A (Fig. 139).

PROBLEM 108.

From a given point a' to project a line contained by a given plane, and making the same angle with the H.P. *as that made by the given plane.* Fig. 140.

Let H.T. and V.T. be the traces of the given plane.

From given point a' draw the perpendicular $a'a$. From a draw the line ab perpendicular to the H.T. Draw the projector bb'. Join $b'a'$. ba is the horizontal, and $b'a'$ the vertical projections of the required line.

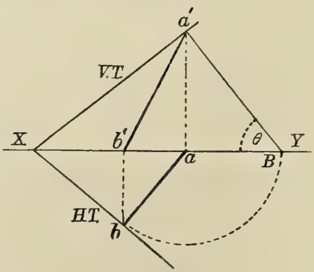


Fig. 140.

To determine the angle ab forms with the H.P. With a as centre, and radius ab, draw an arc till it meets XY in B. Join Ba'. Then θ is the angle made by the plane required.

Problem 109.

To project a line making an angle of 40° *with the* H.P., *and contained by a given plane.* Fig. 141.

Let H.T. and V.T. be the traces of the given plane.

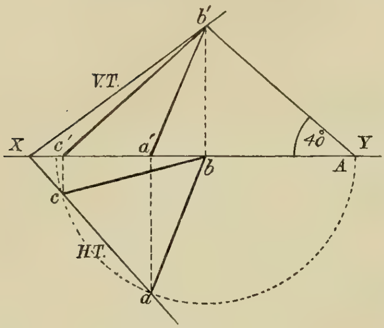


Fig. 141.

At any convenient point A on XY draw Ab' at the required angle (40°) till it meets the V.T. in b'. Draw the perpendicular $b'b$. With b as centre, and radius bA, draw a semicircle. This will represent the base of a semi-cone of which b' is the vertex. This semicircle intersects the H.T. in the points c and a. Join cb and ab, either of which is the horizontal projection of the required line.

Draw the projectors aa' and cc'. Join $a'b'$ and $c'b'$, either of which is the vertical projection of a line fulfilling the required conditions.

Problem 110.

From a given point A *to project a line forming an angle of* 60° *with the* H.P., *and contained by a given plane.* Fig. 142.

Let H.T. and V.T. be the traces of the given plane, and aa' the projection of the given point A.

From a draw ab parallel to the H.T. Draw the projector bb' till it meets the V.T. Draw $b'a'$ parallel to XY. Draw a'C making an angle of

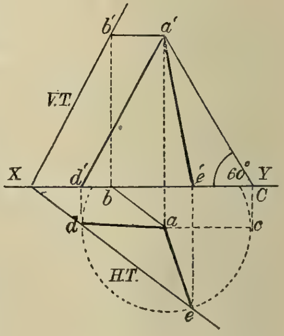


Fig. 142.

60° with XY. Draw the perpendicular C*c* till it meets a line from *a* parallel to XY. With *a* as centre, and radius *ac*, draw an arc till it meets XY. This arc intersects the H.T. in *d* and *e*. Join *da* and *ea*. Draw projectors from *d* and *e*. Join *d′a′* and *e′a′*. *da* and *d′a′* are the projections of one line, and *ea*, *e′a′*, of the other, either of which is contained by the given plane, and is at the required angle with the H.P.

Problem 111.

From a given point A *to project a horizontal line contained by a given plane.*

The lines *ab*, *a′b′* (Fig. 142), are the projections of the line required.

Problem 112.

From a given point A *to project a line making an angle of* 30° *with the* H.P., *and contained by a given plane inclined to both planes of projection, but parallel to* XY. Fig. 143.

Let H.T. and V.T. be the traces of the given plane, and *aa′* the projection of the given point.

Through *a* draw a line at right angles to XY and cutting it in *c*, and the V.T. in *c′*. With *c* as centre, and radii *ca* and *cd*, draw arcs meeting XY in A and *e*. Join *ec′*. Draw a projector from A till it meets *ec′* in A′. Draw A′*a′* parallel to XY. From *a′* draw the line *a′f* making an angle of 30° with XY. With *c* as centre, and radius *cf*, draw an arc till it meets the H.T. in *b*. Join *ab*, which is the horizontal projection of the required line. From *b* draw the projector *bb′*. Join *a′b′*, which is the vertical projection of the line required.

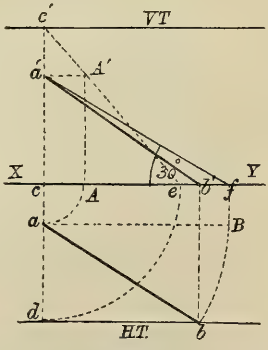


Fig. 143.

Problem 113.

To determine the traces of a plane containing a given horizontal line AB, *and inclined at an angle of* 30° *with the* H.P. Fig. 144.

Let *ab*, *a′b′*, be the projections of the given line.

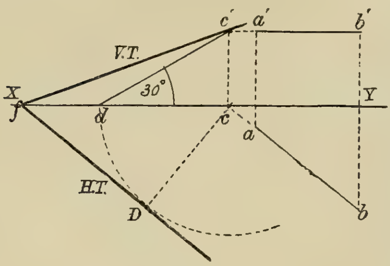


Fig. 144.

Produce *ab* till it meets XY in *c*. Draw a projector to *c* till it meets *a′b′* produced in *c′*. From *c′* draw the line *c′d*, making an angle of 30° with XY. With *c* as centre, and radius *cd*, draw an arc. From *c* draw a perpendicular to *cb* till it meets the arc in D. Through D draw a line parallel to *ab* till it meets XY in *f*. This is the H.T. of the plane required. Join *fc′*, which is the V.T.

Problem 114.

To determine the traces of a plane containing a given line AB *inclined to both planes of projection, the plane to be inclined at an angle of* 60° *with the* H.P. Fig. 145.

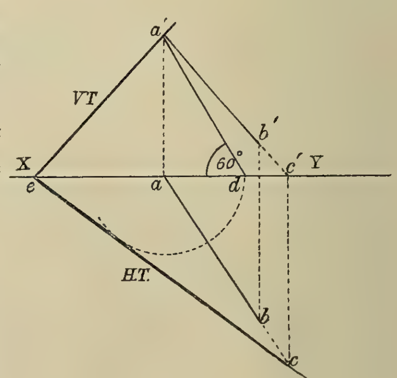


Fig. 145.

Let *ab*, *a′b′*, be the projections of the given line.

Draw the projector *aa′*, and produce *a′b′* till it meets XY in *c′*. Produce *ab* till

it meets a projector drawn from c' in c. From a' draw the line $a'd$, making an angle of 60° with XY. With a as centre, and radius ad, draw an arc. From c draw a line tangential to the arc till it meets XY in e. Join ea'. ec is the H.T., and ea' the V.T. of the plane required.

Problem 115.

To determine the traces of a plane containing two given intersecting lines, AB *and* AC. Fig. 146.

Let ab, $a'b'$, and ac, $a'c'$, be the projections of the given lines.

Produce ab till it meets XY in d. Produce $a'b'$ till it meets the projector from d. This is the V.T. of AB. Produce $a'b'$

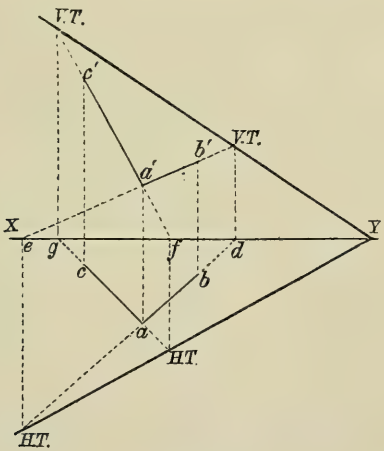


Fig. 146.

till it meets XY in e. Produce ab till it meets the projector from e. This is the H.T. of line AB.

Find the traces of the line AC in the same way. Lines drawn through these traces till they meet on XY are the traces of the plane required.

Note.—The traces of a plane to contain three given points A, B, and C would be found in the same manner.

CHAPTER XXVIII

ROTATION OF PLANE FIGURES

In the projection of various objects the necessity for rotating plane figures at any required angle with the co-ordinate planes is of frequent occurrence. The following problems illustrate the principle of this subject :—

Problem 116.

To project a six-pointed regular star inclined to the H.P. *at an angle of* 45°. Fig. 147.

Let G represent this star, constructed on the H.P.

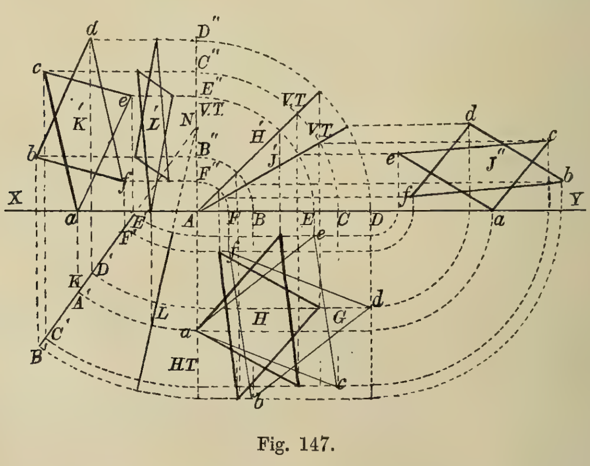


Fig. 147.

Draw a line through point *a* of the figure, perpendicular to XY. Assume this line to be the H.T. of the required projection.

Where this line meets XY in A, set off the required angle, which gives the line H′. This is the V.T. of the required projection.

Draw horizontal lines through the points *a, b, c, d, e,* and *f* of the constructed figure, also perpendiculars till they meet XY in the points A, F, B, E, C, D. With A as centre, and the other points as radii, draw arcs till they meet the V.T. H′, and from these points draw projectors till they meet the horizontal lines drawn through the corresponding points of the original figure. By joining these points as shown, we get the plan H, *i.e.* the original figure is rotated on the H.T. till it is inclined at an angle of 45° with the H.P.

Problem 117.

To project an elevation of the same figure inclined at an angle of 30° *to the* H.P., *its* H.T. *to be parallel to* XY. Fig. 147.

At point A on XY draw the line J′ at an angle of 30° with it. With D as centre, and the points where the horizontal lines drawn through the original figure meet a line drawn through D*d* as radii, draw arcs till they meet XY. From these points on XY draw perpendiculars till they meet horizontal lines drawn from the corresponding points on J′. Join these points as shown, which gives the elevation J″ required.

Problem 118.

Project the same figure in a vertical plane, inclined to the V.P. *at an angle of* 54°. Fig. 147.

From A draw the perpendicular AD″, and assume this line to be the V.T. of the required projection. In any convenient position draw the line B′E′ at the required angle with XY, and produce it till it meets the V.T. in N. This is the H.T. of the required projection.

With N as centre, and the points where the horizontal lines through the original figure meet the line drawn through A*a* as radii,

draw arcs till they meet the H.T. in the points B′, C′, A′, D′, F′, E′. Draw projectors to these points till they meet horizontal lines drawn from corresponding points on the V.T. Join the points as shown, to complete the required projection, of which K is the plan and K′ the elevation.

L and L′ are the plan and elevation of the same figure "rotated" till it is inclined to the V.P. at an angle of 77°.

Note.—The student will see, by studying this illustration (Fig. 147), that we can project a plan or elevation at any possible angle with either co-ordinate plane from a given plane figure, *i.e.* we can assume any H. or V. trace and rotate the figure upon it accordingly.

Problem 119.

To rotate a given equilateral triangle abc (1) *till it forms an angle of* 60° *with the plane of the given figure, and* (2) *till it forms an angle of* 100° *with it.* Fig. 148.

1. Bisect the line *ab* in *d*, and join *cd*. With *d* as centre, and radius *dc*, draw an arc. At *d* let *d*C make the given angle (60°) with *dc* and meet the arc in C. From C draw a perpendicular to *dc* to meet it at *c*′. Join *ac*′ and *bc*′. Then *abc*′ is the triangle required. The triangle *dc*′C is rotated on the line *dc*′ till it is perpendicular to its original position, and *c*′ is the projection of the point C.

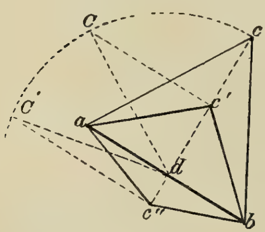


Fig. 148.

2. At *d* let C′*d* make the given angle (100°) with *dc*, meeting the arc in C′. From C′ draw a line perpendicular to *cd* produced in *c*″. Join *c*″ to *a* and *b*. Then *abc*″ is the triangle required.

Problem 120.

To rotate a regular pentagon abcde on a line af drawn through its centre till it forms an angle of 60° *with the plane of the given figure.* Fig. 149.

Join *be*. The lines *be* and *cd*, being at right angles to the axis *af*, represent the traces of planes in which the points *be* and *cd* of the pentagon move. With *g* as centre, and radius *gb*, draw arcs from *b* and *e*. With *f* as centre, and radius *fc*, draw arcs from *c* and *d*.

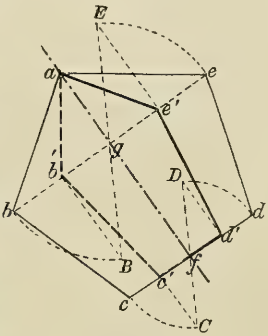


Fig. 149.

At *g* let E*g* make the given angle (60°) with the line *ge*, and produce it till it meets the arcs from *b* and *e* in the points B and E. From these points draw lines perpendicular to *be* till they meet it in the points *b′* and *e′*. Join *a* to *b′* and *e′*. Find the points *c′* and *d′* in the same manner, and join *b′c′* and *e′d′*. Then *a′b′c′d′e′* is the projection required. If the axis *af* were on the H.P., it would be the H.T. of the plane containing the projection, and the dotted half of the figure would be below the H.P.

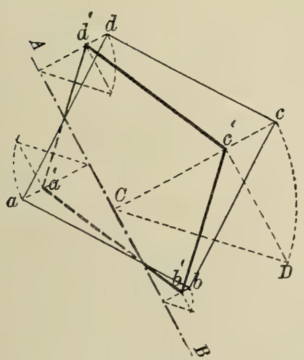


Fig. 150.

Problem 121.

To rotate a given square abcd on an axis AB *till it forms an angle of* 45° *with the plane of the original figure.* Fig. 150.

From each of the four points *a*, *b*, *c*, and *d* draw perpendiculars to the axis AB.

With C as centre, and radius C*c*, draw an arc. At C let C*d* make

an angle of 45° with the line Cc, and produce it till it meets the arc drawn from c in D. From D draw a perpendicular till it meets Cc in c'. Proceed in the same manner with each of the other points, and join them. $a'b'c'd'$ is the projection required. The dotted part of the figure is below the plane of the original figure.

CHAPTER XXIX

OBJECTS INCLINED TO THE H.P. AT GIVEN ANGLES

In the preceding chapter we have inclined the plane of the figure, as a whole, to a given angle, without concerning ourselves with the inclination of any particular line contained by the plane. The problems in this chapter are constructed with additional data, viz. :—

1. The inclination of a line contained by the plane is given in addition to the inclination of the plane itself.

2. The individual inclination of two lines contained by the plane are given.

Problem 122.

To determine the projection of an equilateral triangle ABC *of* $1\frac{1}{4}''$ *sides, its surface to be inclined to the* H.P. *at an angle of* 60°, *and one of its edges* (BC) *at an angle of* 44°. Fig. 151.

At any convenient point D on XY set off an angle of 60° and draw the line DE, which is the V.T. of the required projection. Draw the perpendicular DG, which is the H.T.

At any convenient distance from D on XY fix the position of H, and at this point let HE make an angle of 44° with XY, produce it till it meets the V.T. in E. From E draw E*f* perpendicular to XY. With *f* as centre, and radius *f*H, draw an arc till it meets the H.T. in G. With D as centre, and radius DE, draw an arc till it meets XY in F. Join FG and *f*G. *f*G is the projection of a line inclined at an angle of 44° with the H.P. and contained by a plane inclined at an angle of 60° with

the H.P., consequently it forms one edge of the required triangle. FG is the same line "constructed" on the H.P. From any

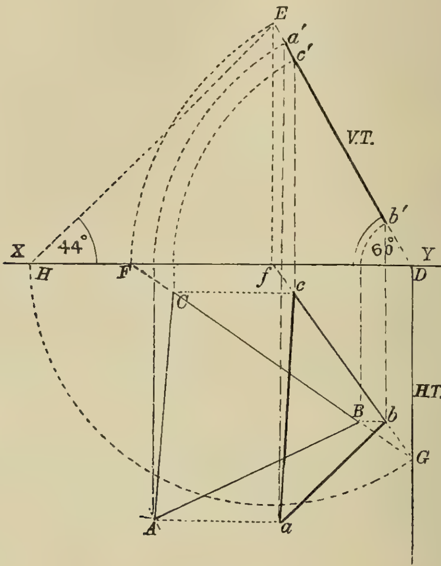


Fig. 151.

convenient point C on FG set off CB $1\frac{1}{4}''$ long, and complete the equilateral triangle ABC.

From each corner of the triangle draw perpendiculars till they meet XY. With D as centre, and these points as distances, draw arcs till they meet the V.T. in the points a', c', and b'. Draw projectors from these points till they meet horizontal lines drawn from the corresponding letters of the triangle, which give the points a, b, c. Join these points for the projection required.

Problem 123.

1. *Project a square of* 6·7″ *sides with its surface inclined to the* H.P. *at an angle of* 50° *and one of its edges at an angle of* 40° *with the* H.P. Fig. 152. *Scale* $\frac{1}{8}$ *full size.*
2. *On the projection of the square complete the plan of a cube.*

3. *Project an elevation of the same cube as seen in the direction of the arrow at* D.

1. Determine the H.T. and V.T. of a plane inclined at an angle of 50° with the H.P.; also the projection and construction of a line in that plane and inclined at an angle of 40° with the H.P. (Prob. 122). *f*G is the projection of this line, and

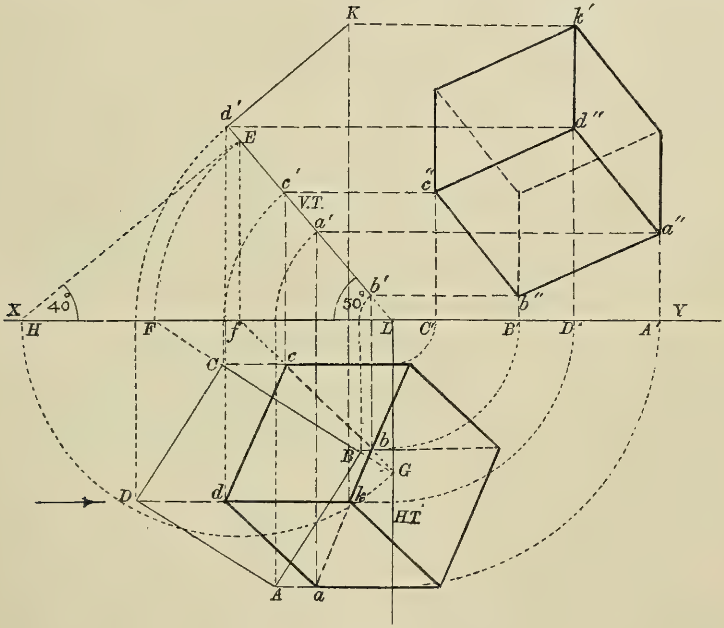


Fig. 152.

FG its construction on the H.P. At any convenient point C on FG set off CB 6·7″ long, and on it complete the square ABCD.

Determine the projections of the points *a*, *b*, *c*, *d* (Prob. 122) and join them, which gives the projection of the square required.

2. From *d′* set off *d′*K perpendicular to the V.T. and equal in length to one side of the square. From K drop a projector till it meets a line drawn from *d*, parallel to XY, in the point *k*. Join *dk*. This gives the projection of one edge of the cube.

As lines of equal length parallel to each other have their projections also equal, draw lines parallel to dk, and equal to it in length, from the points a, b, and c, and join them to complete the plan of the cube.

3. Draw lines from the points a, b, c, d parallel to XY till they meet the H.T. With L as centre, and these points as radii, draw arcs till they meet XY in the points C′, B′, D′, A′. Draw perpendiculars on these points till they meet horizontal lines drawn from the corresponding points on the V.T., which give the points c'', b'', d'', a'', by joining which we get the bottom of the cube.

Produce the perpendicular D′d'' till it meets a horizontal line drawn from K. Draw lines on a'', b'', c'' parallel to $d''k'$, and equal to it in length. By joining the ends of these lines we obtain the elevation required.

Problem 124.

1. *Project a square* ABCD, *of* 0·9″ *side, the side* AB *to be inclined at an angle of* 45°, *and the diagonal* BD 43° *with the* H.P. Fig. 153.
2. *Let the square represent the base of a cube. Complete its plan and elevation.*

Note.—It is not necessary that one corner of the projected square should be in the H.P.

1. With B as centre, and with any assumed height of B above the H.P. as radius, draw an arc. Draw tangents to this arc at angles of 45° with AB and 43° with BD, meeting the lines produced in E and F. Draw a line through E and F, which is the H.T. of the required projection.

Draw XY perpendicular to the H.T. Draw lines from the points A, B, C, D perpendicular to XY, and till they meet it in the points D′, A′, C′, B′.

Draw a line J parallel to XY, and the same height above it as the assumed radius of the arc on plan. With H as centre,

and HB′ as radius, draw an arc till it meets the line J in b'. Join Hb', which is the V.T. of the required projection. With H as centre, and the points D′, A′, C′, B′ as radii, draw arcs till they meet the V.T. in the points d', a', c', b'. From these points drop projectors till they meet lines drawn from the correspond-

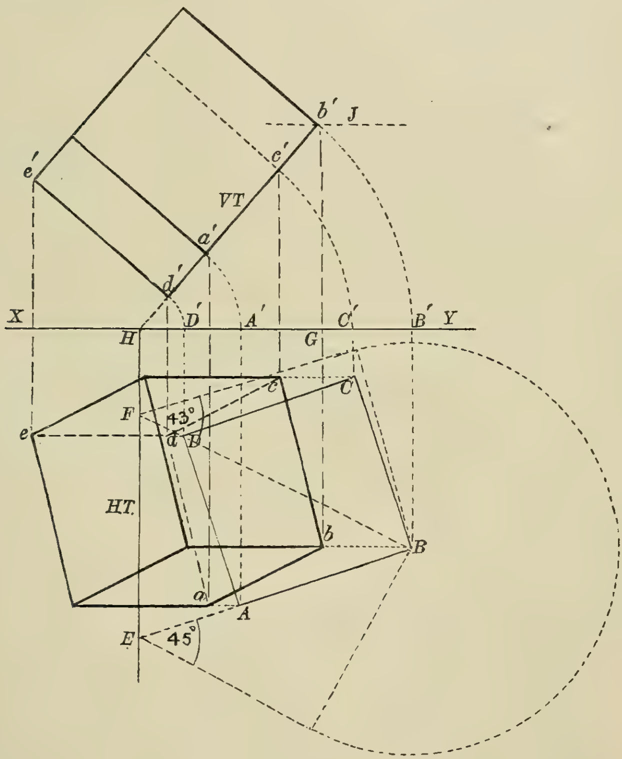


Fig. 153.

ing letters of the square, drawn parallel to XY, which give the points a, b, c, d, by joining which we get the projection required.

2. On the points d', a', c', b' draw lines equal to the side of the square (0·9″) in length, and perpendicular to $d'b'$. Join the tops of these perpendiculars to get the elevation of the cube.

From e' drop a projector till it meets a line drawn from d, parallel to XY, in e. This gives one edge of the cube on plan.

Draw lines from the points *a*, *b*, *c* parallel to *de*, and equal to it in length, and join their ends to complete the plan of the cube.

Problem 125.

1. *Project a regular pentagon* ABCDE, *of* 0·65″ *side, with the side* CD *inclined at an angle of* 44°, *and the diagonal* AC *at* 40° *with the* H.P. Fig. 154.
2. *From this projection draw the plan and elevation of a pentagonal prism with edges* 0·9″ *long.*

1. Assume any convenient height for the point C above the

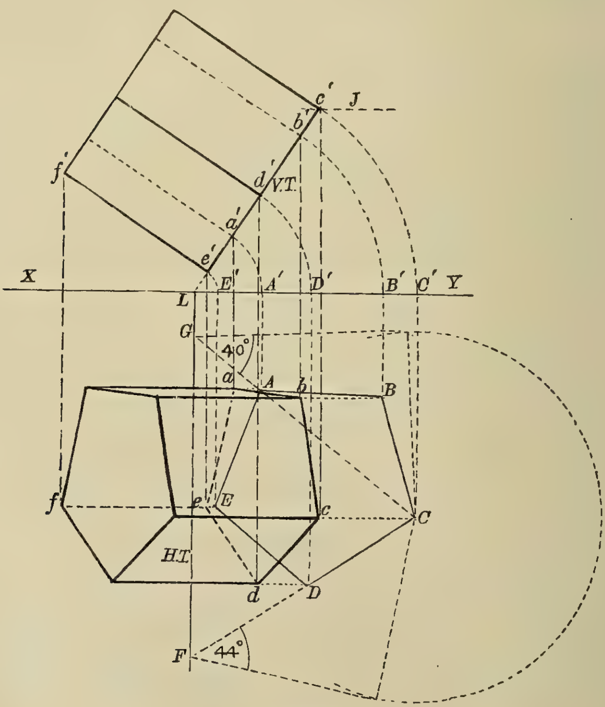


Fig. 154.

H.P., and from C as centre, with this assumed height as radius, draw an arc. Draw tangents to this arc at angles of 40° with

AC and 44° with CD, meeting the lines produced in the points F and G. Draw a line through F and G for the H.T. of the plane of projection. Draw XY perpendicular to the H.T.

Draw lines from the points A, B, C, D, E perpendicular to XY, and meeting it in the points E′, A′, D′, B′, C′. Draw a line J parallel to XY, at a height above it equal to the assumed radius of the arc in plan. With L as centre, and LC′ as radius, draw an arc till it meets the line J in c'. Join Lc' for the V.T. of the required projection. With L as centre, and the points E′, A′, D′, B′ as radii, draw arcs till they meet the V.T. in the points e', a', d', b', c'. Drop projectors from these points on the V.T. till they meet lines drawn parallel to XY from the corresponding points of the pentagon in the points a, b, c, d, e. Join these points, as shown, to obtain the projection required.

2. On the points e', a', d', b', c' draw lines 0·9″ long perpendicular to $e'c'$. Join the tops to complete the elevation.

From f' drop a projector till it meets a line from e drawn parallel to XY in f. From each of the points a, b, c, and d draw lines parallel to ef, and equal to it in length. Join the ends of these lines, as shown, to complete the plan.

CHAPTER XXX

SECTIONS OF SOLIDS, CONSTRUCTION OF SECTIONAL AREAS, AND DEVELOPMENT OF PLANE AND CURVED SURFACES

At the bottom of page 13 a *section* is defined as being the "*intersection* of a solid by a plane." This plane is called the "*cutting plane*," and in the following problems it is given inclined at different angles to both the co-ordinate planes. The surfaces of the solids cut through are projected, and the true shapes of the sectional areas are "constructed." In some instances, where necessary, the exterior surfaces of the solids are "developed" into their true shape.

Problem 126.

To project a cube of $\frac{3}{4}''$ *edge, standing on the* H.P., *and inclined at an angle of* 30° *to the* V.P., *intersected by a cutting plane inclined to the* H.P. *at an angle of* 45°, *and perpendicular to the* V.P. ; *the plane to intersect both the horizontal faces of the cube.* Fig. 155.

Draw the plan *abcd* of the cube, and carry up projectors from the points, $\frac{3}{4}''$ above XY, and join them for the elevation.

Find the traces of the cutting plane (Prob. 80 B).

Where the V.T. cuts the elevation in the points e' and f', drop projectors which will intersect the plan in the lines *eh* and *fg*. *afgche* is the plan of the cut surface of the cube, and $e'a'c'f'$ the elevation.

The sectional surface can be "constructed" by rotating the projected surface of the section on either the H.T. or V.T.

To rotate the plan of the section on the H.T. Draw lines at right angles to H.T. from the points of the plan, and make the lengths of these lines from the H.T. equal to the distances of

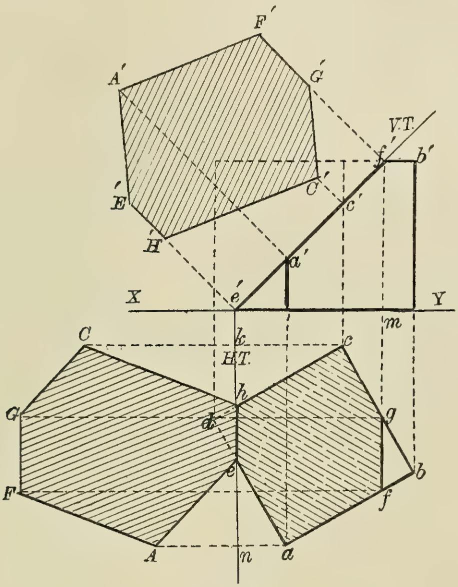


Fig. 155.

the corresponding points on the V.T. from XY ; *e.g.* to obtain the point C, set off *k*C equal to *e′c′*, and so on with each of the other points. Join them, as shown, to complete the construction of the sectional surface.

To rotate the sectional surface on the V.T. Draw perpendiculars from the points *e′*, *a′*, *c′*, *f′*, and make the lengths of these lines equal to the distances of the corresponding points below XY ; *e.g.* make *f′*G′ equal to *mg*, *f′*F′ equal to *mf*, and so on with the other points. Join them as shown.

Problem 127.

To project a quadrilateral prism $10\frac{1}{2}''$ *long, with a base* 6″ *square, standing on its base on the* H.P., *with its longer edges parallel*

to the V.P., *and one of its sides inclined to the* V.P. *at an angle of* 60°. *Intersect the prism with a plane parallel to* XY, *and inclined to the* H.P. *at an angle of* 30°. Scale $\frac{1}{8}$ full size. Fig. 156.

Project the prism (Prob. 3). Find the traces of the cutting plane (Prob. 80 C).

At any convenient point *e* draw the line E*e′* at right angles

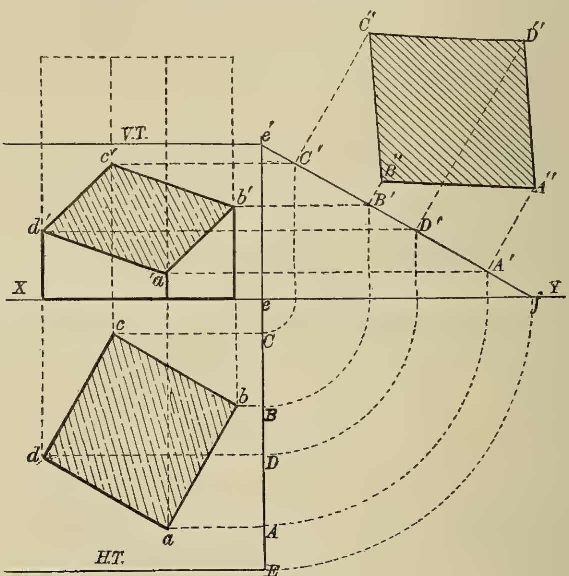


Fig. 156.

to XY, cutting the H. and V. traces. With *e* as centre, and radius *e*E, draw an arc till it meets XY in *f*. Join *e′f*. *efe′* is the angle the cutting plane forms with the H.P.

Draw lines from each point in the plan perpendicular to *e*E, and meeting it in the points C, B, D, A. With *e* as centre, and each of these points as radii, draw arcs; and where they meet XY draw perpendiculars till they meet *e′f* in the points C′, B′, D′, A′. From these points draw horizontal lines till they meet projectors drawn from the corresponding points in the plan in the points *c′*, *b′*, *d′*, *a′*. Join these points for the elevation of the section.

To construct the sectional area. Draw perpendiculars to $e'f$ at the points C′, B′, D′, A′, and make their lengths equal to the distances of the corresponding points in the plan from the line eE; *e.g.* make C′C″ equal to Cc, B′B″ equal to Bb, and so on with the other points, and join them.

Problem 128.

Project a regular hexagonal prism $9\frac{1}{2}''$ *long, with* $3\frac{1}{2}''$ *sides, standing on its base on the* H.P., *with one of its faces inclined to the* V.P. *at an angle of* 58°. *Intersect the prism by a plane inclined at an angle of* 55° *with the* H.P. *and* 46° *with the* V.P., *the plane to cut through the base of the prism. Construct on the* H.P. *the sectional area. Develop the exterior surface of the cut prism. Scale* $\frac{1}{8}$ *full size.* Fig. 157.

Project the prism (Prob. 9).

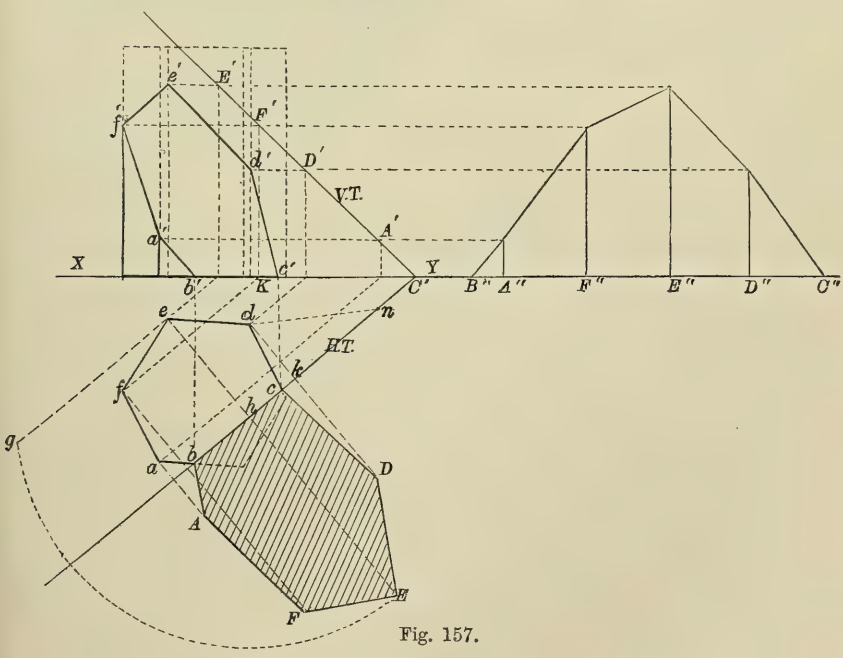


Fig. 157.

Find the traces of the cutting plane (Prob. 83 or 84).

Draw lines from the points of the plan parallel to the H.T. till they meet XY, then draw perpendiculars to XY till they meet the V.T. in the points C′, A′, D′, F′, E′. From these points draw horizontal lines till they meet projectors drawn from the corresponding points of the plan in the points c', d', e', f', a', b', and join them as shown. This is the elevation of the section.

To construct the true shape of the section, we rotate the plan *abcdef* on the H.T. From e draw the line eg parallel to the H.T., and equal in length to the height of e' above XY.

Draw lines from each of the points in the plan at right angles to the H.T. : the one drawn from e will intersect the H.T. in h. With this point as centre, and radius hg, draw an arc till it meets the line from e produced in E. Find all the other points in the same way as shown.

The student should observe that the line hE is the true length of the line he. A simpler method of obtaining the length of these lines is as follows.

Note.—The true length of a projected line is equal to the hypotenuse of a right-angled triangle, the base of which is one of its projections, and the altitude the perpendicular height of the other projection.

We shall understand this better if we refer to Fig. 97. There Abb' is the right-angled triangle, Ab is the base equal to the horizontal projection of the line, and bb' is the perpendicular height of its other projection; so Ab', the hypotenuse, is the true length of the line. We will now find the point D by this method.

Let kd represent the horizontal projection of a line, and Kd' its perpendicular height. From k set off on the H.T. kn equal to Kd'. Then the distance between n and d will be the true length of the line kd. Set this distance off from k on dk produced. This will give the point D. The other points can be found in the same manner.

To develop the surface of the cut prism. Set off on XY the widths of the sides from the plan, and erect perpendiculars till

they meet horizontal lines drawn from the corresponding points in the elevation. Join them as shown.

Problem 129.

Project a regular hexagonal prism $10\frac{1}{2}''$ *long, with* $3\frac{1}{2}''$ *sides, lying on one of its faces on the* H.P., *with its longer edges inclined to the* V.P. *at an angle of* 17°. *Intersect the prism with a plane inclined at an angle of* 50° *to the* H.P. *and* 56° *to the* V.P.; *the plane to intersect all its longer edges. Construct on the* V.P. *its sectional area. Scale* $\frac{1}{8}$ *full size.* Fig. 158.

Project the prism (Prob. 12).

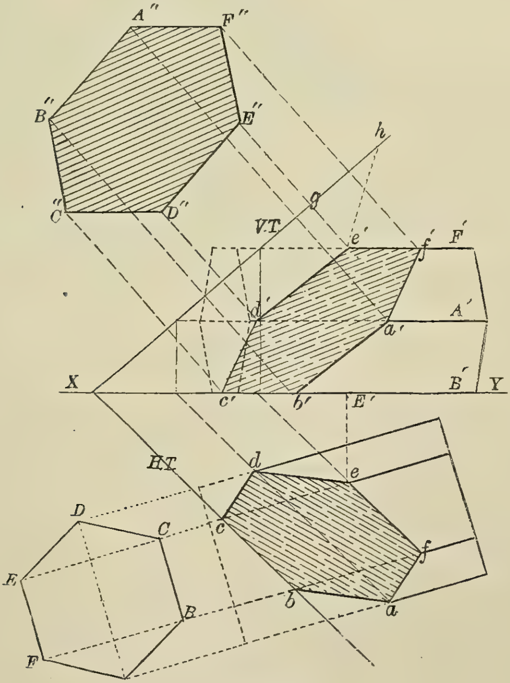


Fig. 158.

Find the traces of the cutting plane (Prob. 83 or 84).

Let BC be the side of the prism resting on the H.P. Join

AD. Then BC, AD, and EF are horizontal lines, and B′A′ and B′F′ their heights above XY.

Note.—A horizontal line contained by a plane would be drawn parallel to the H.T. on plan, and parallel to XY in elevation (see the line $g'h'$, Fig. 118).

As bc is on the H.P., it must coincide with the H.T. Where the lines F′ and A′ produced meet the V.T., drop perpendiculars till they meet XY, and then draw lines parallel to the H.T. Where these lines intersect the lines of the plan they will determine the points of the section. Join them as shown.

Carry up projectors from these points till they meet the corresponding lines in the elevation, and join them.

Note.—The projectors are omitted in several of these problems to save confusion, but the points in plan and elevation bear corresponding letters throughout, so can be easily recognised.

To construct the sectional area on the V.P. Draw lines from each of the points in the elevation at right angles to the V.T. Take the distance of point e below XY as eE′, and set it off on the V.T. from g as gh. Set off gE″ equal to he', the hypotenuse of a right-angled triangle, as described in the preceding problem. Find the other points in the same way, and join them as shown.

Problem 130.

Project a regular pyramid standing on its base on the H.P., *with its sides inclined to the* V.P. *at an angle of* 45°; *the cutting plane to be inclined at an angle of* 43° *to the* H.P. *and* 70° *to the* V.P. *Construct the sectional area on the* V.P. Fig. 159.

Project the pyramid (Fig. 6, C).

Find the traces of the cutting plane (Prob. 83 or 84).

Produce the diagonal eg till it meets the H.T. in E. Draw the projector EE′, and from E′ draw a line parallel to the V.T. Where this line meets the edges of the pyramid in b' and d' will determine two points in the section. Drop projectors from these

points till they meet the corresponding lines of the plan in the points *b* and *d*.

The lines forming the section of this pyramid are really the intersection lines of two planes; the cutting plane being one, and each side of the pyramid the other plane. We know that

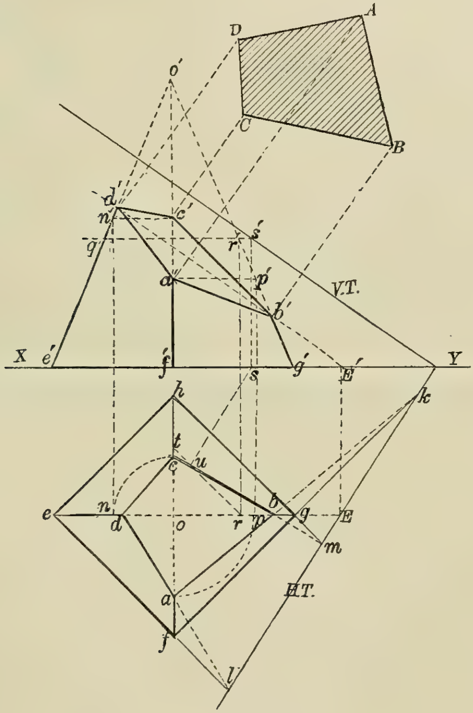


Fig. 159.

the line of intersection between two planes must have its trace where the traces of the two planes intersect (see Fig. 114). Produce three sides of the base of the pyramid till they meet the H.T. in the points *k*, *m*, and *l*. These are the traces of the lines required. Draw a line from *k* through *b*, and produce it till it meets the diagonal *fh* in *a*. Draw lines from *m* and *l* in the same manner till they meet the diagonals in *d* and *c*. Join *dc*. This will complete the plan of the section.

To determine the elevation of the points *a* and *c*. With centre *o*, and radii *oa* and *oc*, draw arcs till they meet the diagonal *eg* in the points *n* and *p*. Draw projectors to these points till they meet the edges of the pyramid in the points *n′* and *p′*. Draw horizontal lines from these points till they meet the other edges of the pyramid in the points *c′* and *a′*. Join the points as shown, to complete the elevation of the section.

Another method of obtaining the section of this pyramid is to assume a horizontal line *q′r′* in any convenient position in the elevation, and drop a projector from *r′* till it meets the diagonal *eg* in *r*. Draw *rt* parallel to the base *gh*. Produce *q′r′* till it meets the V.T. in *s′*. Draw a projector from *s′* till it meets XY in *s*. Draw a line from *s* parallel to the H.T. till it meets the line *rt* in *u*. Then *u* is a point in the plan of the section, which can be completed from the traces *k*, *m*, and *l*, as previously described.

To construct the sectional area ABCD, proceed in the manner described in the preceding problem.

Problem 131.

Project a regular hexagonal pyramid standing on its base on the H.P., *with one of its sides inclined to the* V.P. *at an angle of* 45° ; *the cutting plane to be inclined at an angle of* 42° *to the* H.P. *and* 70° *to the* V.P. *Construct the sectional area on the* V.P., *and on it project another elevation of the pyramid. Develop the exterior surface of the cut pyramid.* Fig. 160.

Project the plan and elevation of the pyramid.

Determine the traces of the cutting plane (Prob. 83 or 84).

Draw the line *gh* through the centre *o* of the plan, at right angles to the H.T. Erect a perpendicular to XY on point *h* till it meets the V.T. in *h′*. Draw the projector *gg′*, and join *g′h′*, cutting the projector from *o* in *o′*. Produce the diagonal through *tq* till it meets XY in *l*. Draw the projector *ll′*, and from *l′* draw a line through *o′*. Where this line cuts the corresponding

edges of the pyramid in elevation to those in plan, it gives the points a' and d'. Produce the diagonal through ru till it meets the H.T. in point m. Draw the projector mm', and from m'

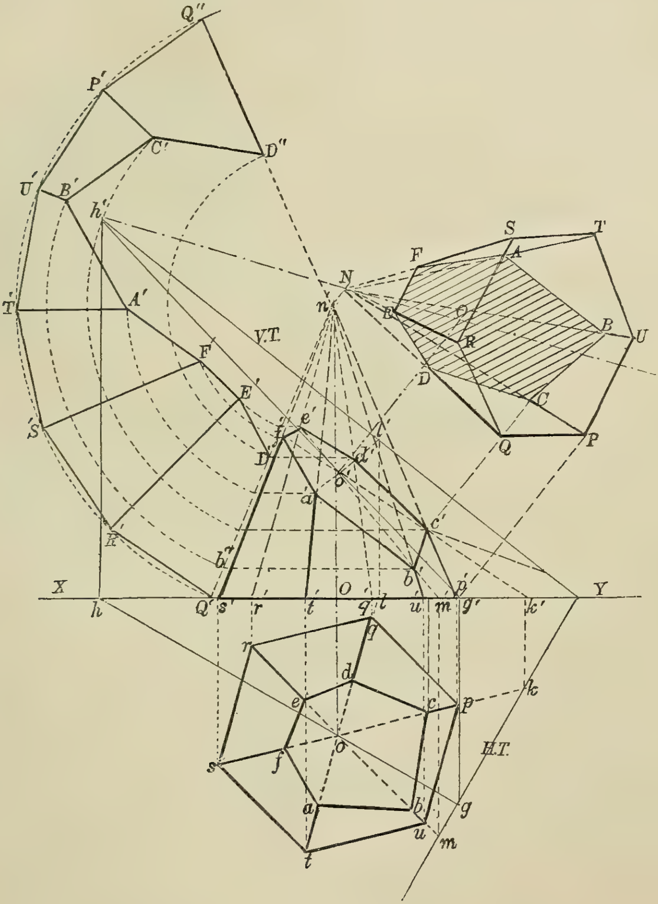


Fig. 160.

draw a line through o' till it meets the corresponding edges of the pyramid in elevation in the points b' and e'. Treat the diagonal sp in the same manner to obtain the points f' and c'. Join these points, as shown, to obtain the elevation of the section.

From each of the points in the elevation of the section drop

projectors till they meet the corresponding edges of the pyramid on plan. Join these points, as shown, to determine the plan of the section.

Construct the sectional area ABCDEF, as described in Prob. 129; also determine the position of centre O in the same manner.

To project another elevation of the pyramid on this "constructed" section. Draw a line from h' through the centre O: this is the axis of the pyramid. Draw a line from n' perpendicular to the V.T. till it meets this axis in N: this is the vertex. Draw a line from p' in elevation, at right angles to the V.T., till it meets a line drawn from N through the point C. This is one point in the base of the pyramid; all the other points are found in the same manner, by joining which we get another elevation of the pyramid.

To develop the exterior surface of the cut pyramid. From O on XY, set off OQ$'$ equal to oq. Join Q$'n'$. Draw horizontal lines through each of the points in the elevation of the section till they meet this line. With n' as centre, and Q$'$ as radius, draw an arc, and step along it six chords, each equal to one side of the base of the pyramid. Join these points, and distinguish them by letters corresponding to those on plan. Draw lines from each of these points towards the vertex n'.

With n' as centre, and b'' as radius, draw an arc till it meets the line drawn from U$'$ towards n'; b' being on the edge $u'n'$ of the pyramid.

Find the other points in the same manner, and join them as shown.

Problem 132.

Project a dodecahedron standing on one of its faces on the H.P., *with one of the edges of its base parallel to the* V.P. *Intersect it by a cutting plane, inclined at an angle of* 45° *to the* H.P., *and an angle of* 60° *with the* V.P. *Construct its sectional area on the* V.P. Fig. 161.

Project the dodecahedron (Prob. 23).

Find the traces of the cutting plane (Prob. 83 or 84).

First determine a cross section through the solid, on the line BC. With C as centre, and all the points of the solid that come on the line BC as radii, draw arcs till they meet XY.

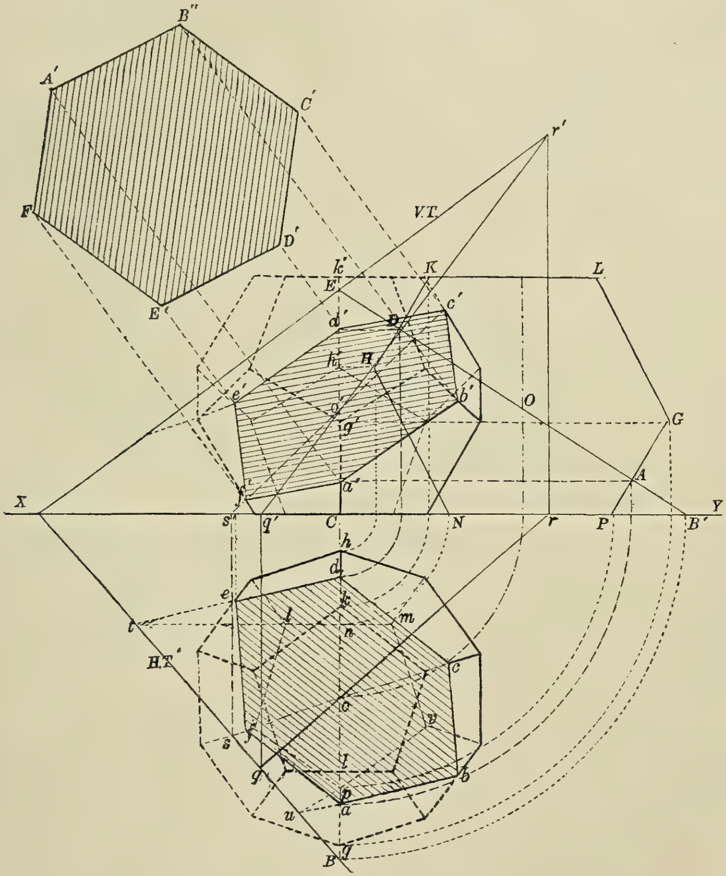


Fig. 161.

Draw perpendiculars at the points on XY till they meet horizontal lines drawn from corresponding points in the elevation; *e.g.* a horizontal line from g' gives point G, one drawn from h' determines point H, etc. By joining these points we get the cross section HKLGPN.

With C as centre, and radius CB, draw an arc till it meets XY in B′. Draw a perpendicular on C till it meets the V.T. in E. Join EB′. This is the angle of the cutting plane on the line BC which cuts the edges PG and HK of the solid in the points A and D. By drawing horizontal lines from these points till they meet the perpendicular on C we get the points a' and d' of the section.

Drop perpendiculars from the points A and D on to XY. With C as centre, and these points as radii, draw arcs till they meet the line BC in the points a and d.

We will now determine where the cutting plane intersects the axis of the solid. To do this we must find the traces of a V.P. perpendicular to the cutting plane containing the axis.

Draw the line qr through the centre o of the plan till it meets XY in r. Draw a perpendicular on r till it meets the V.T. in r'. Draw a perpendicular from q till it meets XY in q'. Join $q'r'$. Then qr is the H.T. of this plane, and $q'r'$ is the line in which it intersects the cutting plane, and o' is the point where the cutting plane intersects the axis.

From where the diagonal through cf meets the H.T. in s draw the projector ss'. Draw a line from s' through o'. This will give the points f' and c' in the elevation of the section, and projectors drawn from these points to the corresponding edges of the plan will give the points f and c.

Produce the edge lm of the base till it meets the H.T. in t. Join dt. This will cut the solid in the point e. Produce the edge vp till it meets the H.T. in the point u. Draw a line from u through a till it meets the solid in b.

Draw projectors to e and b to obtain the points e' and b'. Join all the points, as shown, to obtain the plan and elevation of the section.

Construct the sectional area A′B″C′D′E′F on the V.P., as described in Prob. 129.

PROBLEM 133.

To project a section through a right vertical cone: the cutting plane to be perpendicular to the V.P. ; *to be inclined at an angle of* 45° *to the axis of the cone, but not to intersect its base.* Fig. 162.

This section is an ellipse.

Let DE be the elevation of the section. Divide it into any number of equal parts, *e.g.* six. Draw the axis of the cone, and

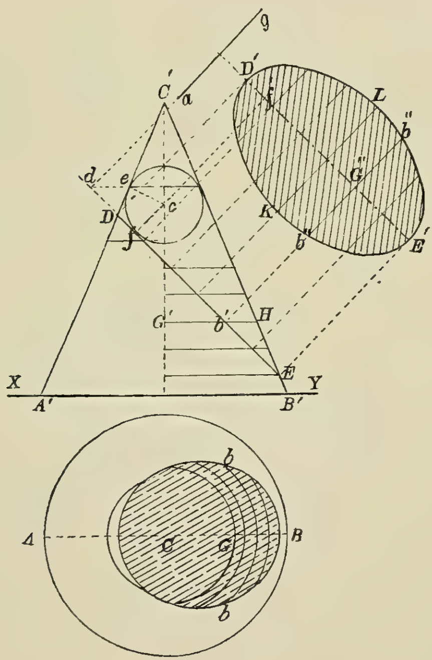


Fig. 162.

through the divisions on DE draw lines parallel to the base of the cone.

The plan of the section is determined by first finding a succession of points in the curve, and then drawing a fair curve through them. We will take the points *bb* as an example.

With C as centre, and a radius equal to G′H (*i.e.* the radius of the cone at the level of b'), draw an arc till it meets a projector drawn from b' in the points bb. Proceed in the same manner with the other points, and draw a fair curve through them.

To construct the sectional area. Draw the line D′E′ in any convenient position, parallel to DE, and draw lines from each of the divisions on DE at right angles to D′E′. Take the distance Gb from plan, and set it off on each side of G″ in the points $b''b''$. Find all the other points in the same manner, and draw a fair curve through them.

Problem 134.

To project a section through a right vertical cone; the cutting plane to be parallel to the side of the cone and perpendicular to the V.P. Fig. 163.

This section is a parabola.

Let D′E′ be the elevation of the section. Divide it into any

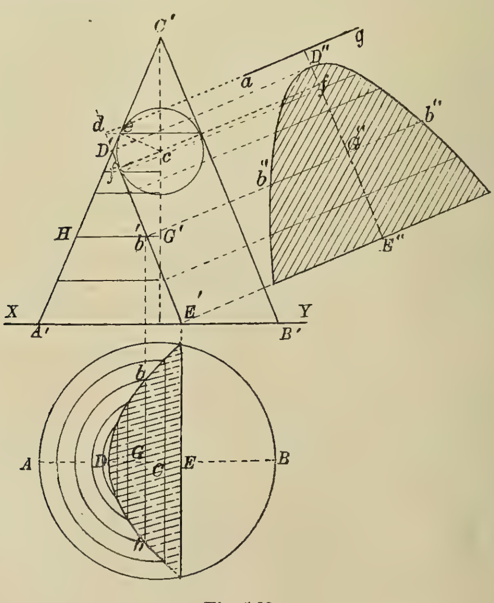


Fig. 163.

number of parts—it is better to have the divisions closer together towards the top. Draw horizontal lines through these divisions.

The plan is determined by finding a succession of points as in the preceding problem. We will take the points *bb* as an example.

With C as centre, and a radius equal to the semi-diameter of the cone at the level of the division *b′*, *i.e.* G′H, draw an arc till it meets a projector from *b′* in the points *bb*. Find the other points in the same manner, and draw a fair curve through them.

To construct the sectional area. Draw the line D″E″ in any convenient position, and draw lines at right angles to it from the divisions on D′E′. Take the distance G*b* from plan, and set it off on each side of G″ in the points *b″b″*. Proceed in the same manner with all the other points, and draw a fair curve through them.

Problem 135.

To project a section through a right vertical cone; the cutting plane to be perpendicular to the H.P., *and inclined at an angle of* 50° *to the* V.P. Fig. 164.

This section is an hyperbola.

Let DE be the plan of the section. From C draw the line C*d* perpendicular to DE. With C as centre, and radius C*d*, draw an arc cutting AB in *n*. Draw the projector *nn′*. *n′* is the vertex of the section. Divide the height *g′n′* into any number of divisions,—they should be made closer together near the vertex,—and draw horizontal lines through them till they meet the sides of the cone.

The elevation of the section is determined by first finding a succession of points, and then drawing a fair curve through them. We will take the points *b′b′* as an example.

With the point C on plan as centre, and a radius equal to G′H (the semi-diameter of the cone at the level of *b′*), draw arcs intersecting the line DE in the points *bb*. Draw projectors to

these points till they meet the line drawn through G′H in the points $b'b$. Find the other points in the same manner, and draw a fair curve through them.

If the cutting plane were perpendicular to both the

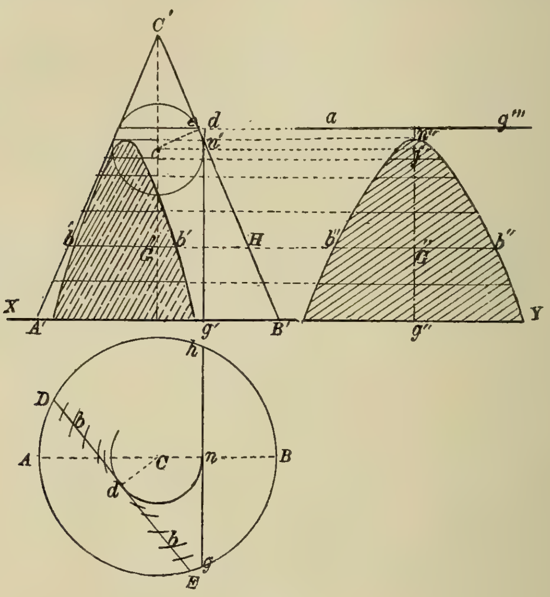


Fig. 164.

co-ordinate planes, gh would be the plan, and $g'n'$ the elevation of the section of the cone.

To construct the sectional area. Draw the line $n''g''$ in any convenient position parallel to the axis of the cone. Produce the divisions on $n'g'$. Take the distance db from plan, and set it off on each side of G″ in the points $b''b''$. Find all the other points in the same way, and draw a fair curve through them.

As the three preceding problems are conic sections, their sectional areas could be constructed by the methods described in Chap. XV. (Plane Geometry), but we must first determine the major and minor axes of the ellipse, and the directrices and foci of the parabola and hyperbola.

We will illustrate by a perspective view (Fig. 165) the principle of the relation between the directrix and focus of a parabola, and afterwards apply it to the ellipse and hyperbola.

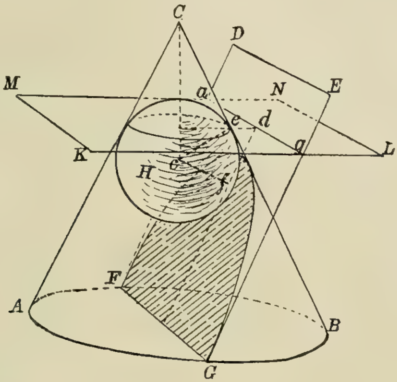


Fig. 165.

ABC is a cone, and DEGF the cutting plane. H is a sphere touching the cutting plane, and inscribed in the upper portion of the cone. A line drawn from *c*, the centre of the sphere, perpendicular to the cutting plane, will meet it in *f*, which is the focus of the parabola.

The plane KLNM, containing the circle of contact between the sphere and the cone, intersects the cutting plane in the line *ag*, which is the directrix of the parabola.

A line joining the centre of the sphere with the circle of contact, as *ce*, is perpendicular to the side of the cone.

Compare this figure with Fig. 295 (Plane Geometry).

Let us now refer to Fig. 162. *cfe* is the inscribed sphere, *f* is the point of contact with the cutting plane, *ce* is perpendicular to the side of the cone, and *e* determines the level of the plane containing the circle of contact. A horizontal line drawn through *e* till it meets the cutting plane produced in *d* will determine the position of the directrix.

Draw a line from *f* perpendicular to the cutting plane till it meets the line D′E′ in *f′* : this is one of the foci of the ellipse.

A line drawn from d perpendicular to the cutting plane will determine the directrix ag.

The line D′E′ is the major axis of the ellipse, and if we bisect this line by another at right angles to it, and obtain the position of the points K and L in the same manner as we determined the points $b''b''$, KL will be the minor axis. We can obtain the other focus and directrix by setting off their distances on the opposite side of KL; or we could construct another sphere in the lower part of the cone, and obtain them as already described.

In Fig. 163, the same construction as previously described will determine the position of the directrix and focus; and as it bears corresponding letters, the student should have no difficulty in understanding it. Compare Fig. 163 with Fig. 165.

The same thing applies to Fig. 164.

Problem 136.

To project the section of a right vertical cone; the cutting plane to be inclined at an angle of 36° *with the* H.P. *and* 73° *with the* V.P., *but not intersecting the base of the cone. This section is an ellipse. Construct the sectional area on the* H.P. *Develop the exterior surface of the cone.* Fig. 166.

Project the cone (Prob. 32).

Find the traces of the cutting plane (Prob. 83 or 84).

First assume a vertical plane perpendicular to the cutting plane passing through the axis of the cone. Draw the line ab perpendicular to the H.T. Draw a perpendicular to XY from a till it meets the V.T. in a'. Draw the projector bb', and join $b'a'$. ab is the H.T. of the V. plane, and $a'b'$ the line in which it intersects the cutting plane.

Where ab intersects the plan of the cone in the points d and e, draw projectors till they meet XY in the points d' and e'.

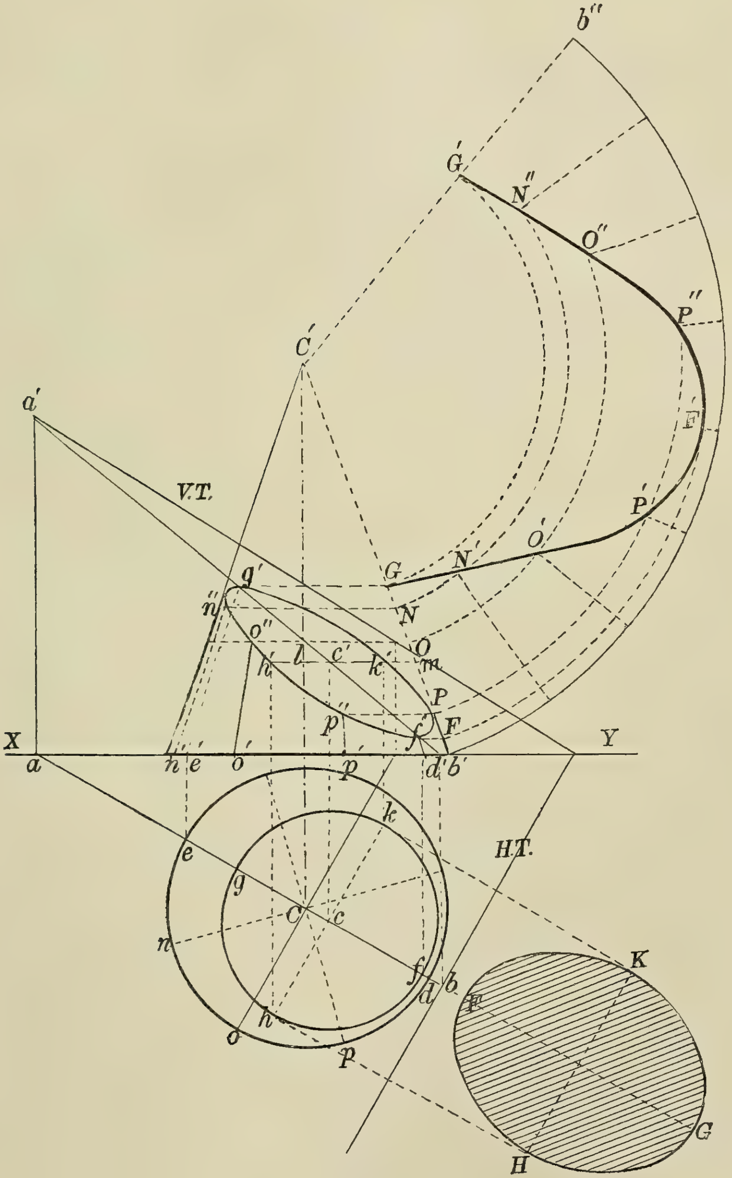

b''
G'
N''
O''
P''
F'
P'
O'
N'
G
C'
a'
V.T.
g'
n''
N
o''
O
h'
l
c'
k'
m
P
p''
f'
F
X
Y
a
n
e
o'
p'
d'
b'
k
H.T.
e
g
C
c
n
f
d
b
F
K
h
o
p
G
H

Fig. 166.

Join these points towards the vertex C′, till they meet the line $a'b'$ in the points f' and g'. Draw projectors from f' and g' till they meet the line ab in f and g. fg is the plan, and $f'g'$ the elevation of the major axis of the ellipse.

Bisect fg in c, and draw a line through it parallel to the H.T. Draw a projector from c till it meets the line $a'b'$ in c'. Draw a horizontal line through c'. To determine the points h and k, *i.e.* the minor axis of the ellipse. We know the minor axis is equal to the diameter of the cone at the level of c', so take the semi-diameter lm and set it off on each side of c, which gives the points h and k. Draw projectors to these points till they meet the horizontal line drawn through c' to determine h' and k'.

Having obtained the projections of the major and minor axes of the ellipse, the sections can be completed by Prob. 181 (Plane Geometry).

To construct the sectional area. Find the points F, H, G, K (Prob. 128), and complete the ellipse (Prob. 181, Plane Geometry).

To develop the exterior surface of the cone. Divide the space from e to d on plan into four equal divisions in the points n, o, p. Draw projectors from these points till they meet XY in the points n', o', p'. Draw lines from these points to the vertex C′ till they meet the section in the points n'', o'', p''.

With C′ as centre, and radius C′b', draw an arc.

Note.—This arc must equal in length the circumference of the base of the cone. It bears the same relation to a complete circle as the radius of the base of the cone bears to the radius of the arc; *e.g.* the radius of the base of the cone in this instance is just one-third of the radius of the arc, therefore the length $b'b''$ is one-third of a circle, and the angle $b'C'b'' = 120°$.

Divide the arc $b'b''$ into eight equal parts. Draw horizontal lines through the points g', n'', o'', p'', f' of the section till they meet the line b'C′. With C′ as centre, and these points as radii, draw arcs till they meet lines drawn from the divisions on the arc $b'b''$ to C′ in the points G, N′, O′, P′, F′, P″, O″, N″, G′. Draw a fair curve through these points.

Problem 137.

To project the section of a right vertical cylinder; the cutting plane to be inclined at an angle of 36° *with the* H.P. *and* 73° *with the* V.P., *but not intersecting the base. This section is an ellipse. Construct the sectional area on the* H.P. *Develop the exterior surface of the cylinder.* Fig. 167.

Project the cylinder (Prob. 35).

Find the traces of the cutting plane (Prob. 83 or 84).

Assume a vertical plane passing through the axis of the cylinder, and perpendicular to the cutting plane. Draw *ab* perpendicular to the H.T. Draw a projector to *a* till it meets the V.T. in *a′*. Draw a projector to *b* till it meets XY in *b′*. Join *a′b′*. *ab* is the V.T. of this V. plane, and *a′b′* the line in which it intersects the cutting plane.

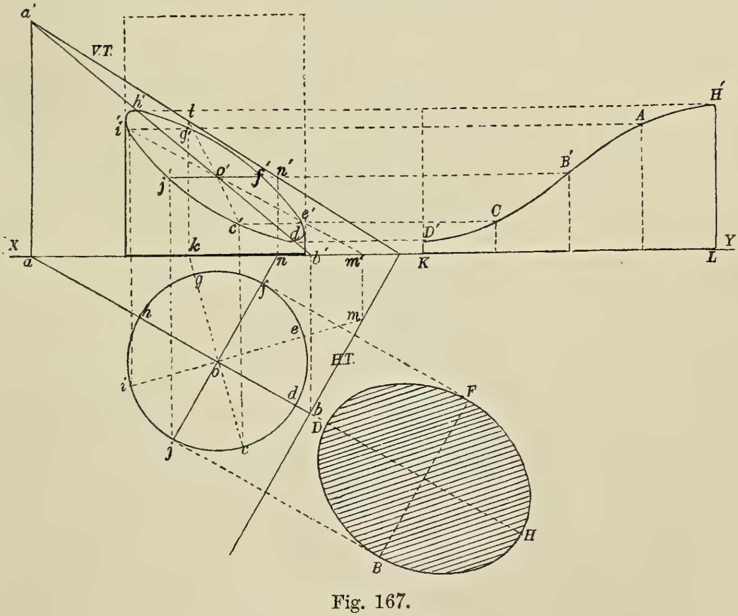


Fig. 167.

Draw a projector to *o* till it meets *a′b′* in *o′*.

Divide the plan by diameters into eight equal parts, one of these diameters, *dh*, being in the H.T. of the V. plane. Projectors to *dh* will give d', h', two of the points in the section. Produce the diagonal *ie* till it meets the H.T. of the cutting plane in *m*. Draw the projector mm', and draw a line from m' through o'. This will give the corresponding points i', e', in the elevation.

Produce the diameter *jf* till it meets XY in *n*. Draw a perpendicular to XY at *n* till it meets the V.T. in n'. Draw a line from n' through o'. This will give the corresponding points j', f' in the elevation. Find the points c', g' in the same manner.

Draw a fair curve through these points for the elevation of the section.

Any number of points in the curve could be found in the same manner by drawing additional diameters to the plan, but eight points are generally deemed sufficient.

To construct the sectional area. Find the points B, H, F, D (Prob. 128), and complete the ellipse (Prob. 181, Plane Geometry).

To develop the exterior surface of the cylinder. Draw the line KL equal to half the circumference of the base of the cylinder (Prob. 192, Plane Geometry), and divide it into four equal parts. Erect perpendiculars to each of these divisions till they meet horizontal lines drawn from the points of the section in the points D′, C, B′, A, H′. Draw a fair curve through these points.

Only one-half of the exterior surface is here developed, to save space; the other half is simply a repetition of the points here found.

Problem 138.

To project the section of a sphere; the cutting plane to be inclined at an angle of 35° *with the* H.P., *and* 74° *with the* V.P. Fig. 168.

Find the traces of the cutting plane (Prob. 83 or 84).

As a sphere is a continuous surface without any edges or angles, it will be necessary to assume certain fixed lines upon its surface in order to determine where the cutting plane will intersect it; meridians and parallels are best suited for this purpose.

Project the sphere with meridians and parallels (Prob. 39).

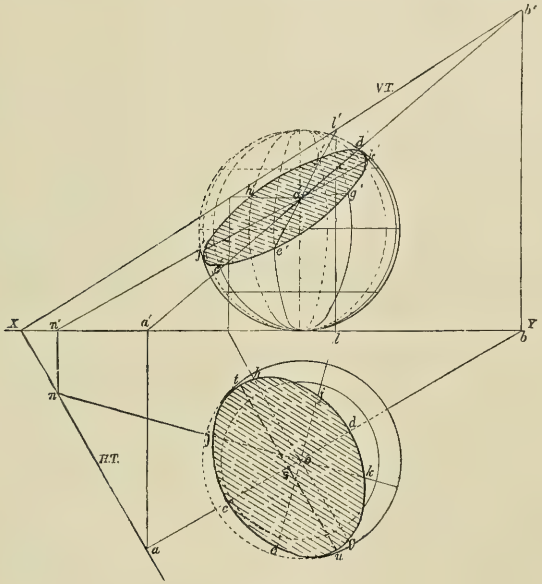


Fig. 168.

It will be better to arrange the meridians on plan so that one of them is parallel to the H.T.

Assume a V.P. perpendicular to the cutting plane and containing the axis of the sphere. Let *ab* be the H.T. of this plane, and *a′b′* the line in which it intersects the cutting plane. Where this line intersects the axis will determine *o′*, and where it intersects the meridian *c′d′* will give two points in the section.

Produce the meridian *jk* on plan till it meets the H.T. in *n*. Draw the projector *nn′*, and draw a line from *n′*, through *o′*, till it meets the meridian *j′k′*. These are two more points in the section.

Produce the meridian *ef* till it meets XY in *l*. Draw a perpendicular at *l* till it meets the V.T. in *l′*. Draw a line from *l′*, through *o′*, till it meets the meridian *f′e′*, which are two more points in the section. Obtain the points *h′* and *g′* in the same manner, and draw a fair curve through all the points found.

Drop projectors from each of these points till they meet the corresponding meridians on plan, and draw a fair curve through them.

These projections are ellipses. They could also be found by first projecting their conjugate diameters, and then completing them as shown by the problems in Chap. XV.

The true shape of the section is of course a circle. To obtain its radius, bisect *cd* in *s*, and draw a line *tu* through *s* parallel to the H.T.; *st* is the radius required; *tu* is the major axis, and *cd* the minor axis of the ellipse.

CHAPTER XXXI

TANGENT PLANES AND SOLIDS

If two spheres touch each other, the line joining their centres will pass through the point of contact.

If a plane touches a sphere, the line joining the point of contact with the centre of the sphere will be perpendicular to the plane.

If two cylinders touch each other, their axes being parallel, they do so in a line; and a plane containing the axes of the cylinders will also contain the line of contact.

A plane tangential to a cylinder is parallel to its axis.

If a plane is tangential to a cylinder, another plane containing the line of contact and the axis of the cylinder will be perpendicular to the tangent plane.

If the surfaces of two cones touch, and their vertices coincide, a plane containing the axes will also contain the line of contact.

A plane tangential to a cone will be perpendicular to a plane containing the line of contact and the axis.

If a sphere touches the side of a cone, the point of contact will be on a plane containing the axis of the cone and the centre of the sphere. The same thing applies to a sphere and cylinder.

Problem 139.

1. *Find the traces of a* V. *plane inclined at an angle of* 45° *to the* V.P., *and tangential to a given vertical cylinder.*

2. *Find the projection of the line of contact.* Fig. 169.

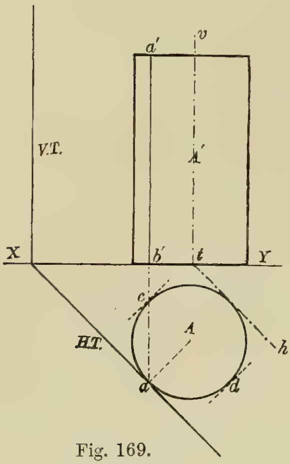


Fig. 169.

1. Let A be the plan, and A′ the elevation of the given cylinder.

Draw the H.T. at an angle of 45° with XY, tangential to the circle A, and where it meets XY draw the V.T. parallel to the sides of the elevation A′.

2. At the point *a*, where the H.T. touches the plan A, draw a projector, which gives the line *a′b′*, the projection of the line of contact required.

Problem 140.

Find the traces and line of contact of a plane tangential to a given horizontal cylinder, whose axis is parallel to the V.P.; *the plane to be inclined at an angle of* 55° *to the* H.P., *and parallel to* XY. Fig. 170.

Let A and A′ be the plan and elevation of the given cylinder.

We will first make an end elevation of the cylinder. Draw the projections of the axis *ef*, and *e′f′*. Draw the line *c*P, and assume it to be the edge of the V.P. Produce XY to T, and assume *c*T to be the edge of the H.P. With *c* as centre, and radius *ce*, draw the arc *e*E. Draw a perpendicular at E till it meets *e′f′* produced in E′. With E′ as centre, and radius E′E, draw a circle. This is the end elevation. Draw the line TP at an angle of 55° to the H.P., and tangential to the circle. This is the edge of the tangent plane, and where it meets the co-ordinate planes in T and P gives the positions of the H. and V. traces in the end elevation. With *c* as centre, and radius *c*T, draw an

arc till it meets a perpendicular from *c* in *t*. From *t* draw the H.T., and from P the V.T., both parallel to XY.

From C′, where TP touches the circle, draw the perpendicular C′C. With *c* as centre, and radius *c*C, draw the arc C*a*. Draw *ab*

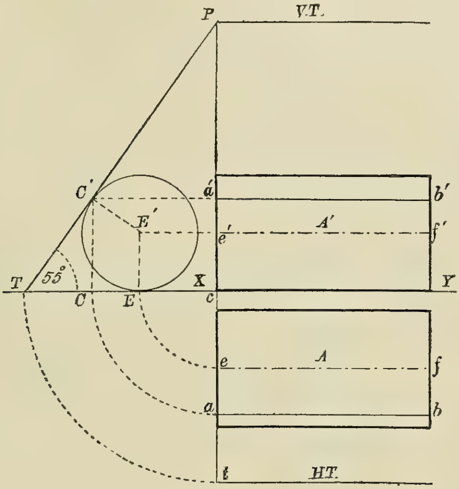


Fig. 170.

parallel to XY. This is the H. projection of the line of contact. From C′ draw a horizontal line, which will give *a′b′*, the V. projection of the line of contact.

Note.—In the following problems, T.P. will signify *tangent plane.*

Problem 141.

To determine the traces and line of contact of a T.P. *to a given horizontal cylinder inclined to the* V.P.; *the* T.P. *to be inclined to the* H.P. *at an angle of* 44°. Fig. 171.

Let A and A′ be the plan and elevation of the given cylinder.

We will first construct an end elevation, as in the preceding problem. Produce the end of the plan, and assume it to be a new ground line X′Y′. Draw the axis *ef*, and produce it to E, making *e*E equal to *eg*. With E as centre, and radius E*e*, draw

a circle. This is the end elevation. At the intersection of the two ground lines, c, draw cP perpendicular to X′Y′. This will represent the edge of the V.P. Draw the line TP at an angle of 44° with X′Y′, tangential to the circle. The points T and P are the positions of the H. and V. traces in the end elevation. Through T draw the H.T. parallel to the axis ef.

We must now "rotate" the point P into its proper position

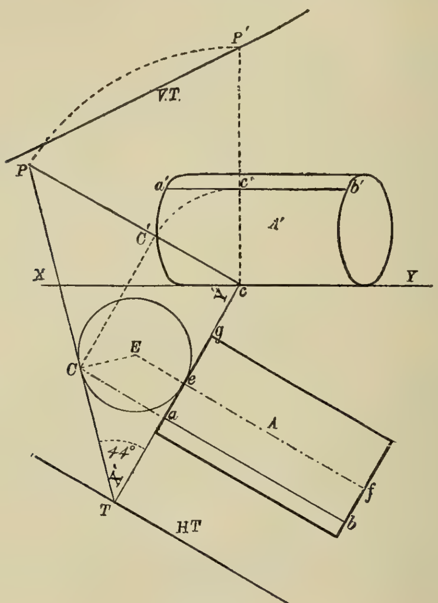


Fig. 171.

on the V.P. To do this, draw cP′ perpendicular to XY. With c as centre, and radius cP, draw an arc till it meets cP′ in P′. This is one point in the V.T. If we were to draw a line through this point to where the H.T. meets XY, it would give us the V.T., but as this point is beyond the limit of the drawing, the line could be determined by Prob. 32 (Plane Geometry).

From point C, where the T.P. touches the circle, draw a line parallel to the H.T. This gives ab, the H. projection of the line of contact.

Draw the line CC′ parallel to X′Y′, till it meets cP in C′. With c as centre, and radius cC′, draw an arc till it meets cP′ in c'. Through c' draw $a'b'$ parallel to XY. This is the V. projection of the line of contact.

Problem 142.

To determine the traces and line of contact of a T.P. *to a given cylinder inclined to the* H.P., *but parallel to the* V.P.; *the* T.P. *to be inclined to the* V.P., *and to pass through a given point.* Fig. 172.

Let A and A′ represent the plan and elevation of the given cylinder, and p' the vertical projection of the given point.

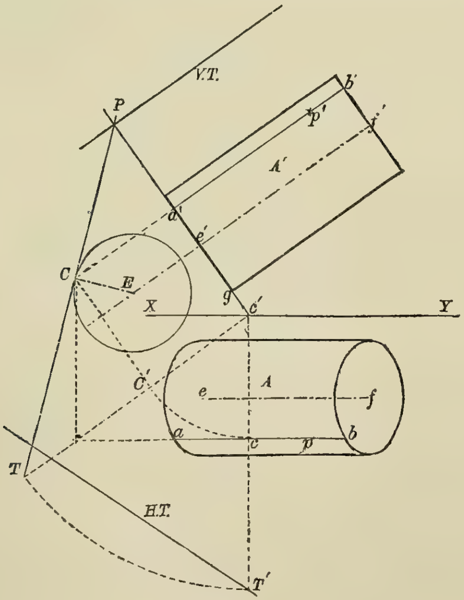


Fig. 172.

Construct an end elevation, assuming a line drawn through ga' to be the edge of the V.P.

Draw the projections of the axis, *ef* and *e′f′*. Produce *e′f′*, and make *e′*E equal to the distance of *ef* from XY. With E as centre, and radius equal to *e′g*, draw a circle. This is the end elevation. Draw a line through *p′*, parallel to the axis *e′f′*, till it meets the circle in C.

Draw a line tangential to the circle at the point C till it meets *ga′* produced in P. Through P draw the V.T. of the T.P. parallel to the axis *e′f′*.

From *c′*, where the assumed edge of the V.P. meets XY, draw a line perpendicular to *c′*P, and meeting PC produced in T. This will represent the edge of the H.P., and the point T will represent the position of the H.T. in the end elevation. We have now to "rotate" this point T into its proper position on the H.P. Draw the line *c′*T′ perpendicular to XY. With *c′* as centre, and radius *c′*T, draw an arc till it meets *c′*T′ in T′. Draw a line through T′, converging towards the point on XY in which the V.T. would meet it if produced (Prob. 32, Plane Geometry).

As the line *a′b′* passes through the given point *p′*, it must be the V. projection of the line of contact.

Draw the line CC′ perpendicular to T*c′*. With *c′* as centre, and radius *c′*C′, draw an arc till it meets *c′*T′ in *c*. Draw *ab* through *c*, parallel to *ef*. Then *ab* is the H. projection of the line of contact.

Note.—In the preceding problems of this chapter the traces for one T.P. only are shown, but there are two tangential planes to each problem, which would fulfil the required conditions. Let us refer to Fig. 169, in which the lines *ht* and *tv* are also the traces of a plane similar to the one explained in the problem. In this special instance, traces could be determined for four tangent planes in accordance with the given data; *e.g.* planes at the required angle could touch the cylinder in the places shown by the points *c* and *d* on plan.

Problem 143.

To determine the traces and line of contact of a T.P. *to a given cylinder inclined to both co-ordinate planes; the* T.P. *to be parallel to a given line.* Fig. 173.

Let A and A′ represent the projections of the given cylinder, and *ab*, *a′b′*, the projections of the given line.

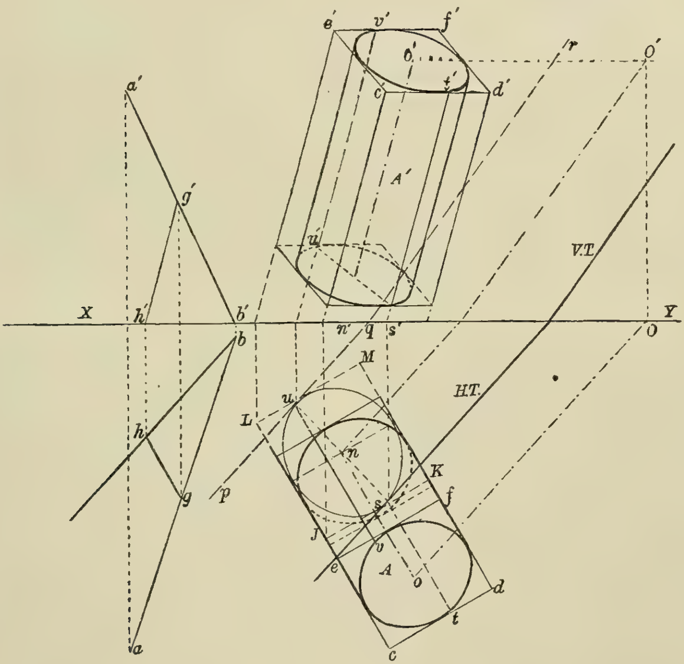


Fig. 173.

First inscribe the cylinder in a rectangular solid. To do this, draw the tangent *cd* on plan perpendicular to the sides of the cylinder produced. Draw another tangent *ef* parallel to *cd*. Then *cdfe* will be one end of the rectangular solid on plan; and if we were to draw diagonals to it, one end of the axis

of the cylinder would be determined. Proceed in the same manner with the opposite end of the cylinder to complete the plan.

To find the elevation. Draw projectors from each of the points *c*, *d*, *f*, *e*, till they meet horizontal tangents drawn to the top of the cylinder, in elevation, in the points c', d', f', e'. Join $e'c'$ and $f'd'$. This will give the upper end of the rectangular solid, and diagonals to it will determine one end of the axis. Proceed in the same manner with the lower end to complete the elevation.

Assume any point *g* on *ab*, and find the projections *gh*, $g'h'$ of a line parallel to the axis of the cylinder. The H. trace of this line will be *h*; and as *b* is the H. trace of the given line, a line drawn through these two points must be the H. trace of a plane containing the given line *ab*, and a line parallel to the axis. It has been already stated that "a T.P. to a cylinder is parallel to its axis," therefore the H. trace of the required T.P. must be parallel to the line *hb*.

Find where the rectangular solid will meet the H.P. This will give the rectangle JKML, by bisecting the sides of which we obtain the major and minor axes of an ellipse. Construct the ellipse (Prob. 181, Plane Geometry). This is the H. trace of the cylinder. Draw the H.T. parallel to *hb*, and tangential to the ellipse.

To determine the V.T. Find the H. trace, *n*, of the axis. Draw *n*N parallel to *hb*. Draw the line *o*O also parallel to *hb*. At O erect a perpendicular till it meets a horizontal line drawn from o' in O′. Join NO′. This is the V. trace of a plane containing the axis. Draw the V.T. parallel to NO′.

The dotted lines *pq* and *qr* would be the traces of another T.P. fulfilling the same conditions as the one already explained.

To find the line of contact. Draw the projector ss'; then make *st*, $s't'$, parallel to the axes. The lines *uv*, $u'v'$, are the projections of a line of contact for a corresponding T.P. on the opposite side of the cylinder.

Problem 144.

To determine the traces of a plane inclined at an angle of 75° *to the* H.P., *and tangential to a given cylinder inclined to the* H.P. *at an angle of* 60°, *and parallel to the* V.P. *Also determine the line of contact.* Fig. 174.

Note.—The inclination of the T.P. cannot be less than that of the cylinder.

Let A and A′ be the projections of the given cylinder. Draw the projections of the axis.

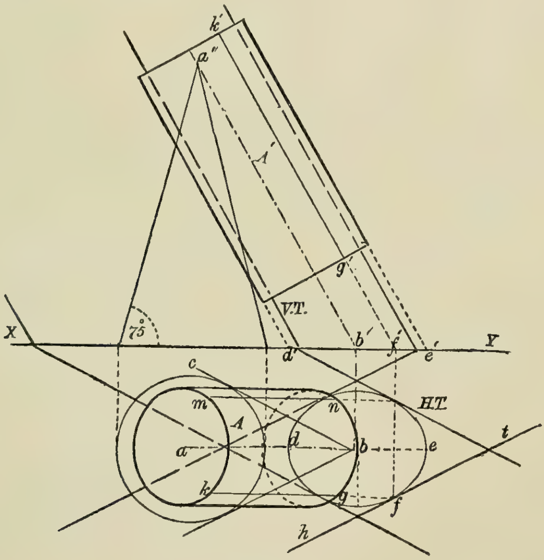


Fig. 174.

Assume any point a' on V. projection of the axis as the vertex of a cone the sides of which form the given angle (75°) with the H.P. Draw the plan and elevation of this cone. Find the H. trace, b, of the axis of the cylinder. Draw the line bc tangential to the base of the cone. Then bc is the H. trace of a plane at the given angle (75°), and containing the axis; so the H.T. of the required plane must be parallel to it.

Produce the sides of the elevation till they meet XY in the points d' and e'. Projectors drawn from these points till they meet the axis on plan will give the points d, e, the major axis of an ellipse. The minor axis is determined by drawing a line through b at right angles to de. Complete the ellipse (Prob. 181, Plane Geometry).

Draw the H.T. parallel to bc and tangential to the ellipse. As the axis of the cylinder is parallel to the V.P., the V. trace must be parallel to the axis; so draw the V.T. parallel to the axis of A′.

As in Fig. 169 there are four tangential planes that fulfil the required conditions, the H.T.s of these planes are shown. Some of the V.T.s are beyond the limits of the drawing, but they are all parallel to the axis of A′.

As there are four T. planes there must be four lines of contact. We will take the one belonging to the plane the H.T. of which is shown by the line ht on plan. This touches the ellipse in f. Draw the projector ff'. Then draw fk, $f'k'$, parallel to the axes. These give the lines gk, $g'k'$, the projections of the line of contact. The line of contact for the T.P., the traces of which are marked H.T. and V.T., has its H. projection in the line mn. Its V. projection will be behind the line $g'k'$.

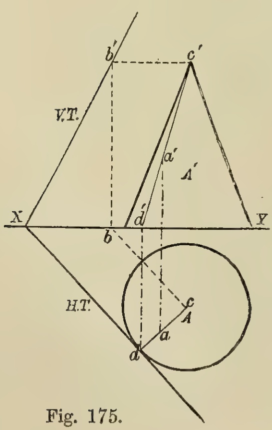


Fig. 175.

Problem 145.

To determine the traces and line of contact of a T.P. *to a given right cone; to touch the cone in a given point.* Fig. 175.

Let A and A′ be the projections of the given cone, and a, a', of the given point.

Draw the lines cd, $c'd'$, through the given point. Draw the H.T. tangential to the circle A through the point d.

Draw the line *cb* parallel to the H.T. From *b* draw a perpendicular till it meets a horizontal line drawn from *c′* in *b′*. Draw the V.T. through *b′* till it meets the H.T. on XY.

The lines *cd, c′d′*, are the projections of the line of contact.

Problem 146.

To determine the traces and line of contact of a T.P. *to a given right cone; the* T.P. *to pass through a given point outside the cone.* Fig. 176.

Let A and A′ be the projections of the given cone, and *a, a′*, of the given point.

Draw lines from the projections of the vertex through *aa′*. These give the H. trace of the line at *e*. Draw the H.T. through *e*, tangential to the circle A.

Draw the line *cb* parallel to the H.T. At *b* erect a perpendicular till it meets a horizontal line drawn from *c′* in *b′*. Draw the V.T. through *b′* till it meets the H.T. on XY.

Draw *cd* perpendicular to the H.T. Draw the projector *dd′*, and join *d′c′*. Then *dc, d′c′*, are the projections of the line of contact.

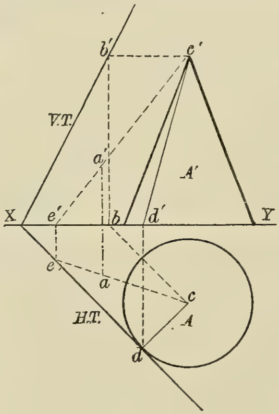


Fig. 176.

Problem 147.

To determine the traces and line of contact of a T.P. *to a given right cone, inclined to the* H.P., *but with its axis parallel to the* V.P.; *the* T.P. *to be parallel to a given line.* Fig. 177.

Let A and A′ be the projections of the given cone, and *ab*, *a′b′*, of the given line.

Draw the projections of the axis.

Produce the sides of the cone in elevation till they meet XY in the points d', e'. Let fall projectors from d' and e' till they meet the axis in plan in the points d and e. This is the major axis of an ellipse. To determine the minor axis, bisect the line de in f, and draw gk at right angles to it. Through k

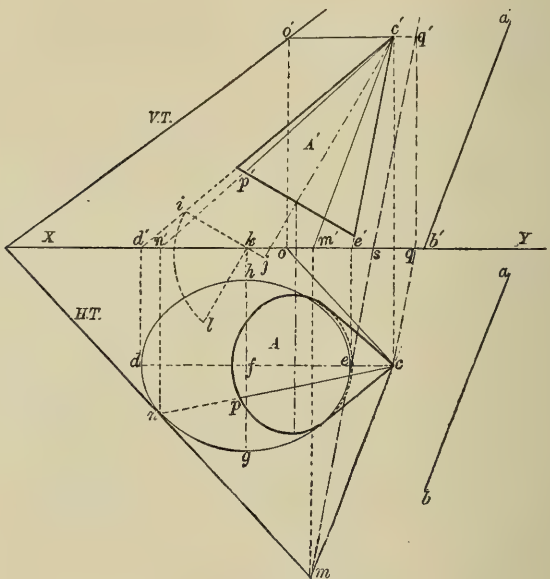


Fig. 177.

draw ij perpendicular to the axis produced. With j as centre, and radius ji, draw an arc till it meets a line drawn from k parallel to the axis in l. From f set off on gh, fg and fh, each equal to kl. Then gh is the minor axis of the ellipse. Draw the ellipse (Prob. 181, Plane Geometry). This ellipse is the H. trace of the cone.

From c and c' draw lines cm, $c'm'$, parallel to the lines ab, $a'b'$. These give the H. trace, m, of a line parallel to the given line, and containing the vertex of the cone.

From *m* draw a line tangential to the ellipse at *n*. This is the H.T. of the required plane.

From *c* draw *co* parallel to the H.T. At *o* erect a perpendicular till it meets a horizontal line drawn from *c′* in *o′*. Draw a line through *o′* till it meets the H.T. in XY. This is the V.T. of the plane required. Another T.P. fulfilling the same conditions could be drawn. Its traces are shown by the dotted lines *ms* and *sq′*.

To find the line of contact. Draw the projector *nn′*. Join *nc* and *n′c′*. Then *pc* and *p′c′* are its projections.

Problem 148.

To determine the traces and line of contact of a T.P. *to a given right cone, the axis of which is inclined to the* H.P. *at an angle of* 53°, *but parallel to the* V.P.; *the* T.P. *to be inclined to the* H.P. *at an angle of* 80°. Fig. 178.

Note.—The inclination of the T.P. must not be less than that of the upper surface of the cone.

Let A and A′ be the projections of the given cone.

Draw the projections of the axis.

Assume *c′* to be the vertex of a right vertical cone, the sides of which make the given angle (80°) with the H.P. Draw the plan and elevation of this cone.

Produce the sides of the elevation of the given cone till they meet XY in the points *d′* and *e′*. Let fall projectors from these points till they meet the axis on plan in the points *d* and *e*. This gives the major axis of an ellipse.

To determine the minor axis. Bisect *de* in *f*, and draw *gk* at right angles to it. Through *k* draw *ji* perpendicular to the axis produced. With *j* as centre, and radius *ji*, draw an arc till it meets a line drawn parallel to the axis in *l*. From *f*, on the line *gk*, set off *fg* and *fh*, each equal to *kl*. *gh* is the minor axis.

Construct the ellipse (Prob. 181, Plane Geometry). This ellipse is the H. trace of the inclined cone.

Four T. planes can be drawn in accordance with the given data: any line tangential to the ellipse and the base of the assumed vertical cone is a H. trace of one of these planes. Four such lines are shown on plan: two touching the ellipse and

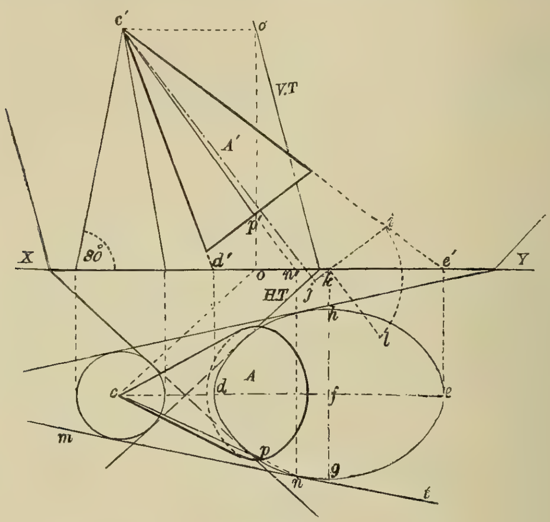


Fig. 178.

circle internally, and two externally. We will take the one marked H.T., and find its corresponding V.T. From c draw co parallel to H.T. Draw a perpendicular to XY at o till it meets a horizontal line from c' in o'. Draw a line through o' till it meets H.T. on XY. This is the V.T. The other V. traces could be found in the same way.

As there are four T. planes, there are also four lines of contact. We will take the one belonging to the T. plane that has its H.T. marked mt. This touches the ellipse at n. Draw the projector nn'. Join nc and $n'c'$. These lines give pc and $p'c'$, the projections of the line of contact on the cone.

The others can be found in the same manner.

Problem 149.

To determine the traces of a T.P. *to a given sphere, touching a given point in the upper part of the sphere.* Fig. 179.

Let A and A′ be the plan and elevation of the given sphere, and *a* and *a′* the projections of the given point.

Draw a horizontal line through *a′* meeting the circle in *d′*. Join *d′* to the centre of the sphere *c′*. Through *d′* draw a line perpendicular to *d′c′* till it meets XY in *f′*, and a perpendicular to XY through *c′*, in C. By repeating this construction on the opposite side of A′, we get the elevation of a cone, with C*g* as axis. This cone is tangential to the sphere, and the line *d′a′* is the elevation of the circle of contact between them; so if we find the traces of a plane tangential to this cone, and passing through the given point, they must also be the traces of the plane required.

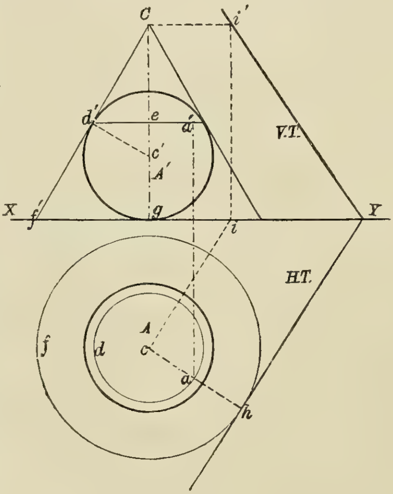


Fig. 179.

With *c* as centre, and radius equal to *ed′*, draw a circle. This is the plan of the circle of contact between the sphere and the cone.

With *c* as centre, and radius equal to *gf′*, draw a circle. This is the plan of the base of the cone. Draw a line from *c* through *a* till it meets this circle in *h*.

Draw H.T. perpendicular to *ch*.

To find the V.T. Draw *ci* parallel to H.T. At *i* draw a perpendicular till it meets a horizontal line from C in *i′*.

Draw V.T. through *i′* till it meets the H.T. on XY.

Problem 150.

To determine the traces of a T.P. *to a given sphere; the point of contact to be in the lower part of the sphere.* Fig. 180.

Let A and A′ be the projections of the sphere, and a' the elevation of the given point of contact.

Draw a horizontal line through a' till it meets A′ in d. Join d to the centre c'. Draw hg through c', perpendicular to XY.

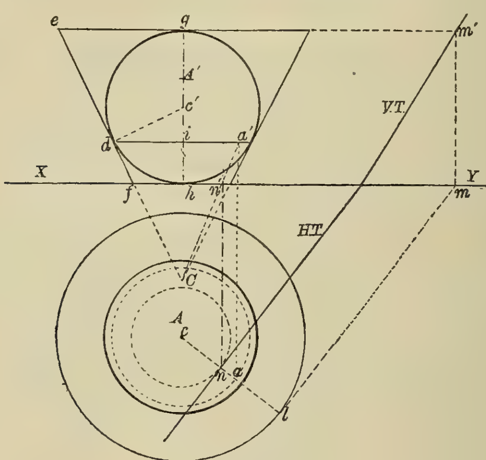


Fig. 180.

Draw a line perpendicular to dc' till it meets XY in f, and a horizontal line from g in e. By repeating this construction on the opposite side of A′ we obtain the elevation of the *frustrum* of an inverted cone tangential to the sphere, da' being the elevation of the circle of contact between them. With c as centre, and radius equal to id, draw a circle. This is the plan of the circle of contact, and a projector drawn from a' gives the plan of the point of contact at a.

With c as centre, and radius equal to hf, draw a circle. This is the H. trace of the *frustrum* of the cone.

Produce the sides of the *frustrum* till they meet at C the

vertex of the inverted cone. Join a' to C, cutting XY in n'. Draw the projector $n'n$. Join cn.

Draw the H.T. perpendicular to cn.

To determine the V.T. With c as centre, and radius equal to ge, draw a circle. This is the plan of the inverted cone. Produce cn to meet this circle in l. Draw lm parallel to H.T. Draw a perpendicular at m till it meets a horizontal line drawn from g in m'.

Draw the V.T. through m' till it meets the H.T. on XY.

Problem 151.

To determine the traces of a T.P. *to a given sphere and cone; the point of contact to be in the lower part of the sphere.* Fig. 181.

Let A and A′ be the projections of the given sphere, and B and B′ the projections of the given cone.

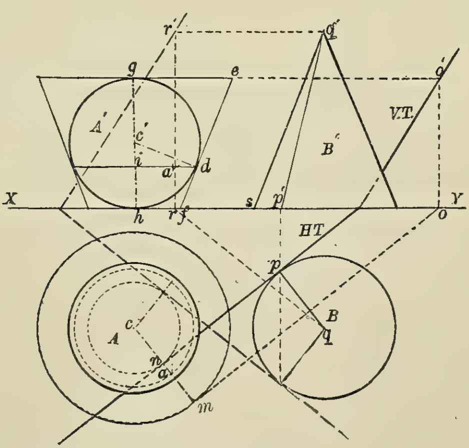


Fig. 181.

This problem is very similar to the preceding one. As the point of contact is to be in the lower part of the sphere, we enclose the sphere in the frustrum of an inverted cone.

Draw the line fe tangential to A′ and parallel to sq'. From the point d, where it touches the circle, draw a line parallel to XY. This line is the elevation of the circle of contact. Draw the line hg through the centre of A′, perpendicular to XY. Draw a line through g parallel to XY, till it meets fe in e. Repeat this construction on the opposite side of A′, to obtain the elevation of the frustrum of the cone.

With c as centre, and radius equal to hf, draw a circle. This is the H. trace of the frustrum. Draw the H.T. of the plane between this circle and B, tangential to both.

Join c to the point n, where the H.T. of the plane touches the H. trace of the frustrum.

With c as centre, and radius equal to ge, draw a circle. This is the plan of the base of the inverted frustrum. Produce cn till it meets this circle in m. From m draw mo parallel to the H.T. At o draw a perpendicular to XY till it meets a horizontal line drawn from g in o'.

Draw the V.T. through o' till it meets the H.T. on XY.

To find the point of contact on the sphere.

With c as centre, and radius equal to id, draw a circle. Where this circle cuts the line cm in a is the plan of the point of contact. A projector drawn from this point till it meets the horizontal line id will give its elevation, a'.

From where the H.T. touches B in p, draw a line to q. This is the plan of the line of contact.

Draw the projector pp'. Join p' to q'. This is the elevation of the line of contact.

There is another T.P. that fulfils the required conditions of this problem; its traces are shown by thick dotted lines in the diagram, as well as the construction for obtaining its contact with the two solids. The V.T. of this plane could be found in the same way as the other, but an alternate method is shown, viz.:—Draw the line qr parallel to the (dotted) H.T. Erect a perpendicular at r till it meets a horizontal line from q' in r'.

Draw the (dotted) V.T. through r' till it meets the corresponding H.T. on XY.

Note.—If the point of contact had been required in the upper part of the sphere, the T. planes would not have passed between the solids, as in the example shown, but on the same side of each solid ; viz. one T.P. would touch both solids in front, while the other would touch them behind.

Problem 152.

The traces of a given T.P. *being given, and the projections of the centre of a sphere, to determine the sphere and point of contact.* Fig. 182.

Let H.T. and V.T. be the traces of the given T.P., and c, c', the projections of the centre of the sphere.

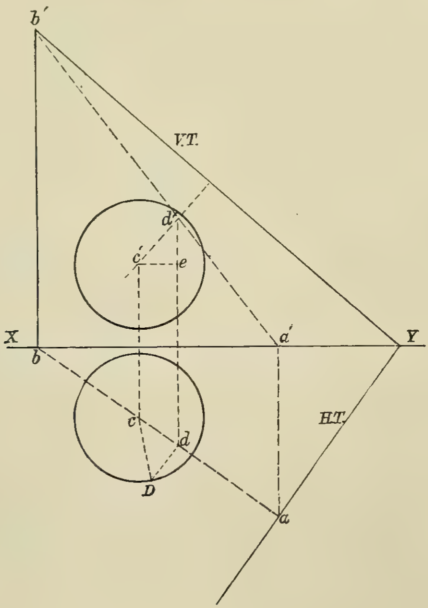


Fig. 182.

Draw the line ab through c, perpendicular to the H.T.

Draw the projector aa', and a perpendicular to XY on b till it meets the V.T. in b'. Join $a'b'$.

Draw a line from c' perpendicular to V.T., cutting $a'b'$ in d'. Draw the projector $d'd$. Then d, d', are the projections of the point of contact.

To determine the radius of the sphere. Draw the horizontal line $c'e$. $c'ed'$ is a triangle, and if we obtain the true length of the hypotenuse $c'd'$, it will give the radius required. To obtain this, we will "rotate" this triangle on plan till it is horizontal. We must bear in mind that the plan of the point e coincides with the point d. Draw dD perpendicular to ab, and equal in length to ed', in elevation. Join cD, which is the radius of the sphere required.

Problem 153.

To determine the traces and points of contact of a T.P. *to two given unequal spheres; the* T.P. *to be inclined at an angle of* 68° *to the* H.P. Fig. 187.

There are four T. planes that could be drawn in accordance

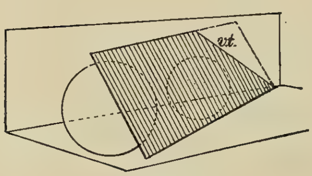


Fig. 183.

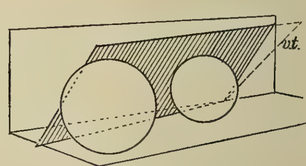


Fig. 184.

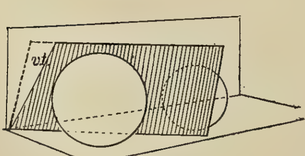


Fig. 185.

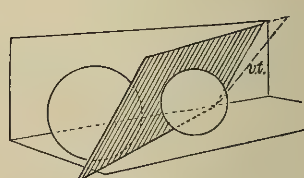


Fig. 186.

with these data. They are illustrated in the perspective views

Figs. 183, 184, 185, and 186. We will first solve the problem with the T.P. in the position illustrated in Fig. 183.

Let A and A′, B and B′, be the projections of the two given spheres.

Envelop each sphere in a right vertical cone; the generatrices of each cone to make the required angle (68°) with the H.P.

Draw the axes *bc′* and *de′*. With *c* as centre, and radius equal to *bf*, draw a circle. With *e* as centre, and radius equal

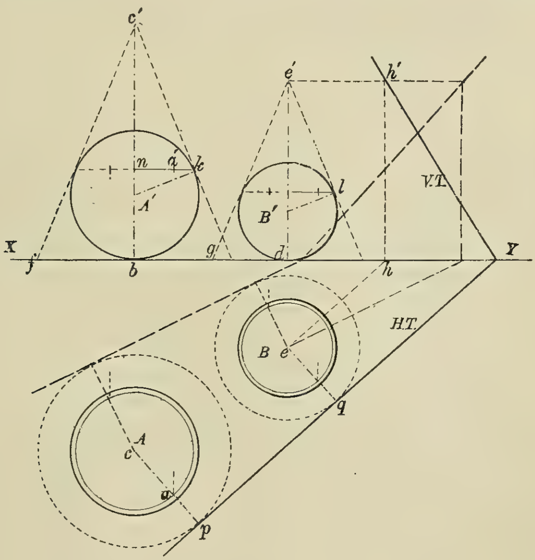


Fig. 187.

to *dg*, draw another circle. These are the plans of the bases of the two cones.

Draw H.T. tangential to these two circles.

Draw *eh* parallel to H.T. At *h* draw a perpendicular to XY till it meets a horizontal line from *e′* in *h′*.

Draw the V.T. through *h′* till it meets the H.T. on XY.

To obtain the points of contact. At *k* and *l*, where the cones touch the spheres, draw horizontal lines. With *c* as centre, and radius equal to *nk*, draw a circle. Draw the line *cp* perpen-

dicular to H.T., cutting this circle in *a*. Draw a projector to *a* till it meets *nk* in *a′*. Then *a* and *a′* are the projections of the point of contact. Proceed in precisely the same manner to find the corresponding points on B and B′.

The traces of the plane illustrated in Fig. 184 are shown by the thick dotted lines. The construction for obtaining these, as well as the points of contact, is shown in the diagram.

To determine the traces of the T.P. in the positions illustrated in Figs. 185 and 186, one sphere in each instance would be enveloped in the frustrum of an inverted cone, as shown in Prob. 151.

Problem 154.

The traces of a plane and the axis of a cone being given, to determine the projections of the cone and its line of contact with the given plane. Fig. 188.

Let H.T. and V.T. be the traces of the given plane, and *ab*, *a′b′*, the projections of the given axis.

First determine a line perpendicular to the given plane, and passing through any point in the axis, as follows:—

Draw *cd* perpendicular to the H.T., and find its elevation *c′d′* by drawing projectors from *c* and *d*. Draw a line from *b′* perpendicular to the V.T. till it meets *c′d′* in *e′*. Draw a projector from *e′* till it meets *cd* in *e*. *be, b′e′*, are the projections of the line required.

Draw the line *bg* perpendicular to *ab*. Draw a projector from *g* till it meets a horizontal line drawn from *b′* in *g′*. Draw *g′h* perpendicular to the line *a′b′*, cutting the V.T. in *i′*. Draw the line *hk* perpendicular to the line *ab*. Then *hk* and *hg′* are the H. and V. traces of the plane of the base of the cone.

Draw projectors from *k* and *i*, giving the points *k′* and *i′* Join *k′i′* and *ki*. Produce *ae* and *a′e′* till they meet these lines in *l* and *l′*. Then *al* and *a′l′* are the projections of the line of contact between the cone and the given plane. They represent

the intersection with the given plane of a plane perpendicular to it, containing the given axis.

The lines bl and $b'l'$ are the projections of the radius of the base of the cone. To determine the true length of this radius, draw lL perpendicular to bl and equal in length to $b'f$, *i.e.* the

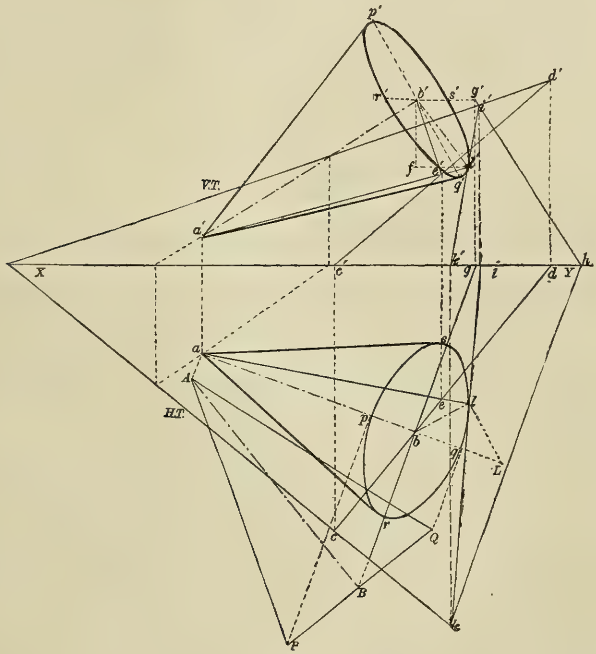


Fig. 188.

vertical height of b' above l'. Join bL, which is the radius required.

Next assume ab to be a ground line. Draw aA and bB perpendicular to it, and equal in length to the heights of a' and b' above XY. Join AB. Draw a line through B perpendicular to AB, and make BP and BQ each equal to the radius bL. Join A to P and Q. Set off from b, on the line bg produced, br and bs, each equal to BP. Draw lines from P and Q perpendicular to ab, and meeting it in the points p and q. Then rs is the major, and

pq the minor axes of the ellipse forming the base of the cone. Complete the plan of the cone (Prob. 181, Plane Geometry).

Draw projectors from *r* and *s* till they meet a horizontal line drawn through b' in r' and s'. Draw a projector from *p*, and make the height of p' above XY equal to *p*P. Find q' in the same way. Join $p'q'$. Then $p'q'$ and $r's'$ are the conjugate diameters of an ellipse forming the base of the cone in elevation. Draw the ellipse (Prob. 185, Plane Geometry), and complete the cone.

Problem 155.

The traces of a plane containing a line being given, to determine the projections of a cone which shall touch the plane in the given line; the vertex angle of the cone to be 45°. Fig. 189.

Let H.T. and V.T. be the traces of the given plane, and *ab*, $a'b'$, the projections of the given line.

Find the projections of any point E on the given line, as ee', and through them draw lines perpendicular to the traces of the plane. Find the H.T. of the line drawn through *e*, as *d*. Join *db*, which is the H.T. of a plane perpendicular to the given plane through the given line. This plane must contain the axis of the cone.

Construct the triangle *bed* on to the H.P. as follows :—Draw a line from *e* perpendicular to *bd* and cutting it in *c*. Draw *e*E′ perpendicular to *ec*, and equal in length to the height of e' above XY. With *c* as centre, and *c*E′ as radius, draw an arc till it meets *ec* produced in E. Join E to *b* and *d*. At *b* draw F*b*, making an angle with *b*E equal to half the given vertex angle ($45° \div 2 = 22\frac{1}{2}°$), and meeting *d*E produced in F. Draw a line from F parallel to *e*E till it meets *de* produced in *f*. Draw a line from *b* through *f*. This is the plan of the axis, indefinite in length.

Draw a projector from *f* till it meets de'' produced in f'.

Draw a line from b' through f'. This is the elevation of the axis, indefinite in length.

Construct the axis on to the H.P. by drawing fF′ perpendicular to fb, and equal in length to the height of f' above XY. Draw a line from b through F′.

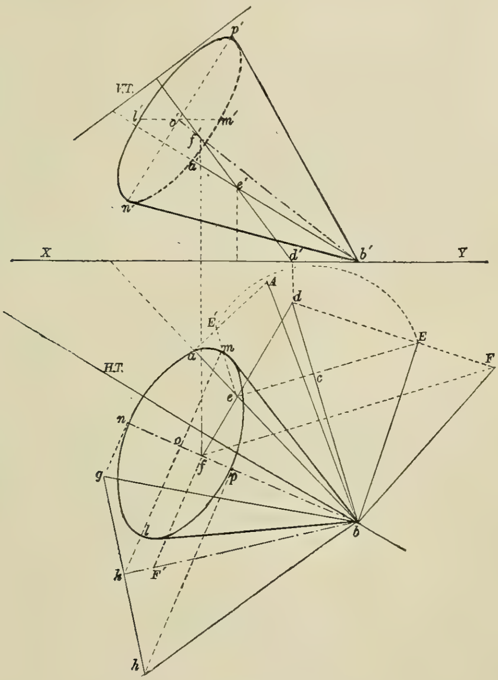


Fig. 189.

Find the true length of ab as follows:—Draw aA perpendicular to ab, and equal in length to the height of a' above XY. Join Ab.

Construct on each side of bF′ an angle equal to half the given vertex angle, and make the lines forming the angles each equal in length to Ab, which give the points g and h. Join gh, cutting bF′ produced in k. Then bk is the true length of the axis.

Draw a line from k perpendicular to bf produced, and cutting it in o. Set off from o on this line, m and l, each equal to kg. From g and h draw lines perpendicular to bo produced, and cutting it in the points n and p. Then ml is the major, and np the minor axes of an ellipse forming the base of the plan of the cone. Draw the ellipse (Prob. 181, Plane Geometry), and complete the cone.

Draw a projector from o till it meets $b'f'$ produced in o'. Draw projectors from l and m till they meet a horizontal line drawn through o' in l' and m'. Find the point n' by drawing a projector from n till its height above XY is equal to ng. Find the point p' in the same way, and join $n'p'$. Then $l'm'$ and $n'p'$ are conjugate diameters of an ellipse. Draw the ellipse (Prob. 185, Plane Geometry), and complete the elevation of the cone.

Problem 156.

To determine the point of contact between a right cone and a sphere: the cone to be lying on its side on the H.P. *with its axis parallel to the* V.P.; *the sphere to be resting on the* H.P. *and to touch the cone* 4″ *from its vertex. Scale* $\frac{1}{8}'' = 1$ *foot.* Fig. 190.

Draw the elevation A′ of the cone, and from it project the plan A. Draw the axis to each projection.

Set off from c' along the upper part of the cone, $c'b$, the required length (4″). Draw ab at right angles to the axis. Draw bE perpendicular to the upper surface of the cone, and equal in length to the radius of the sphere.

With E as centre, and radius Eb, draw a circle. The point of contact required must be on the line ab, as this is the elevation of the circle of contact round the cone at the required distance from its vertex. Produce Eb to d', and draw Ef parallel to ab.

We will now roll the sphere along ab till it rests on the H.P., its required position. When the sphere is resting on the H.P.

we know that the height of its centre above the H.P. is equal to its radius, so draw a horizontal line e' at this height above XY.

Let us imagine the line d'E to revolve round the point d' as a centre, and assume Ef to be the edge of a plane in which E moves. When E reaches the horizontal line e' it gives the centre of the sphere in its required position. Draw the sphere.

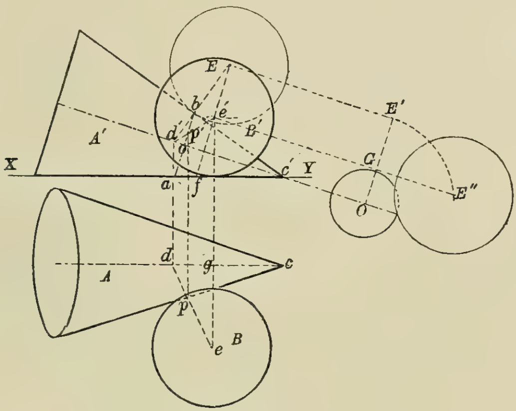


Fig. 190.

To complete the plan we shall require a supplementary elevation of the sphere.

Take a section through the cone, at right angles to its axis, on the line ab.

Produce the axis, and at any point O as centre, and radius equal to ob, draw a circle. Draw OE′ perpendicular to the axis produced. Draw a line from E parallel to the axis till it meets OE′ in E′. With O as centre, and radius OE′, draw an arc till it meets a line from e' parallel to the axis in E″. This is the position of the centre of the sphere. Draw the circle representing it.

Draw a projector from e', and make ge equal to GE″. Then e is the centre of the sphere on plan. Draw it.

We have already stated that the line ab represents the circle of contact between the two solids, so where the line $d'e'$ intersects this line in p' is the elevation of the point of contact.

Draw the projector $d'd$. Join de. Draw the projector $p'p$. Then p is the plan of the point of contact.

PROBLEM 157.

To determine the traces of a T.P. *to three given unequal spheres, tangential to each other, resting on the* H.P.; *the two larger spheres to have their centres on a line parallel to the* V.P. *Project the points of contact between each sphere and the* T.P., *also where the spheres touch each other.* Fig. 191.

Let A, B, and C be great circles of the given spheres.

Take a height above XY equal to the radius of the sphere A, and draw a short horizontal line a'. Draw another horizontal line at b', equal to the radius of B, above XY. At any convenient point a' as centre, and radius equal to the radii of A + B, draw an arc intersecting the line through b' at b'. Draw the circles A′ and B′ about the centres a' and b'.

Draw a horizontal line ab at any convenient position below XY, and draw projectors from a' and b' till they meet it in a and b. Draw the circles A and B about the centres a and b.

We will now make a supplementary elevation (C″) of the smallest sphere. With a' as centre, and radius equal to the radii of A + C, draw an arc. Also with b' as centre, and radius equal to the radii of B + C, draw another arc intersecting the other at C″. Draw the sphere C about this centre C″. Draw a line $d'e'$ parallel to XY, and at a height above it equal to the radius of C, intersecting the arcs just drawn in d' and e'.

Produce ab, and draw projectors from d' and e' till they meet it in d and e. With a as centre, and radius ae; and with b as centre, and radius bd, draw arcs intersecting at c. Draw the sphere C about c as centre. This completes the plan.

Draw a projector from c till it meets $d'e'$ in c'. This is the centre of the sphere C′. Draw it, to complete the elevation.

If we now assume two cones lying with their vertices on the

H.P., and each enveloping two of the spheres, a line joining their vertices must be the H.T. of the T.P. required.

Draw tangents to the circles A and B till they meet in *f*. Also draw tangents to the circles B and C till they meet in *g*. *f* and *g* are the vertices of the assumed cones, and a line joining

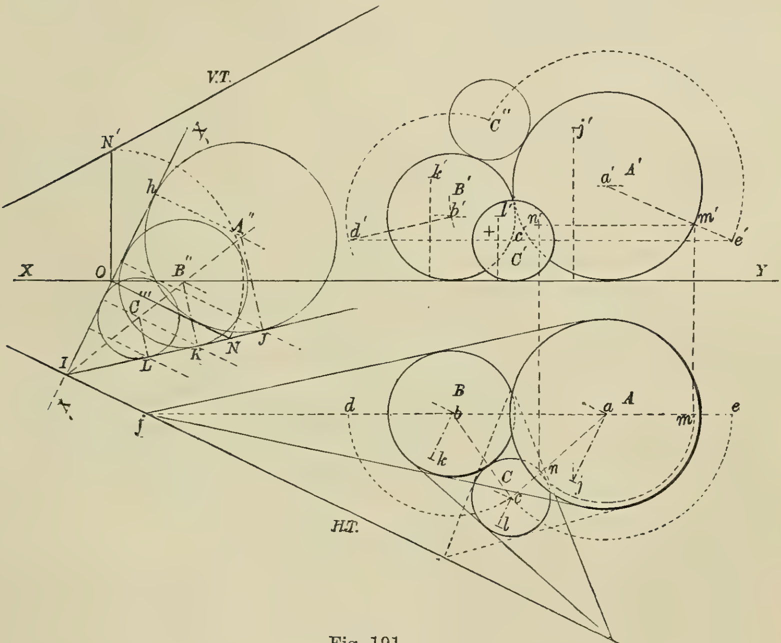


Fig. 191.

them determines the H.T. If we were to draw tangents to the circles A and C, they would also meet on this line, as shown in plan.

We will now draw a supplementary elevation of the three spheres, projected on a V.P. perpendicular to the H.T.

Assume, in any convenient position, a new ground line X′Y′ perpendicular to H.T. Draw a line from *a*, parallel to H.T., till it meets X′Y′ in *h*. Set off on this line *h*A″ equal to the radius of A, and complete the circle with A″ as centre. Join A″ with I, the point where the H.T. meets X′Y′. Also draw the tangent JI to the circle A″.

Where the two ground lines intersect at O, draw ON perpendicular to X′Y′. ION is a right-angled triangle, and if we were to "rotate" it on the line IO till the point N was immediately above O, N would be a point in the V.T. To do this, draw ON′ perpendicular to XY. With O as centre, and radius ON, draw an arc till it meets ON′ in N′. Draw the V.T. converging towards the point where the H.T. meets XY (Prob. 32, Plane Geometry).

Draw lines from b and c parallel to the H.T. till they meet A″I in the points B″ and C‴. These are the centres of the spheres B and C in the supplementary elevation. Draw them.

Draw A″J, B″K, and C‴L, each perpendicular to the tangent JI. Draw lines from the centres a, b, and c perpendicular to the H.T., till they meet lines drawn from the points J, K, and L parallel to H.T. in the points j, k, and l. These are the points of contact of the T.P. with each sphere on plan.

To obtain corresponding points in the elevation, draw projectors from the points j, k, l on plan, and make their heights above XY equal to the heights of the corresponding points J, K, and L above X′Y′. Then j', k', and l' are points of contact between the T.P. and the spheres, in elevation.

To determine the point of contact between the spheres A and C. Join ac. Draw the projector $m'm$. With a as centre, and radius am, draw an arc till it meets ac in n. This is the plan of the point of contact required.

Draw a projector from n till it meets a horizontal line drawn from m' in n'. This is the elevation of the point of contact between the spheres A and C. The points of contact between A and B, and B and C, are determined in the same manner.

Problem 158.

To determine the traces and point of contact of a T.P. *to a given sphere; the* T.P. *to contain a given line.* Fig. 192.

Note.—There are two T. planes that fulfil these conditions:

one touches the upper surface of the sphere, and the other its lower surface.

Let A and A′ be the plan and elevation of the given sphere, and *ab*, *a′b′*, the projections of the given line.

Find the H. and V. traces of the given line, as C and D′ (Prob. 67).

If we assume two enveloping cones, tangential to the sphere,

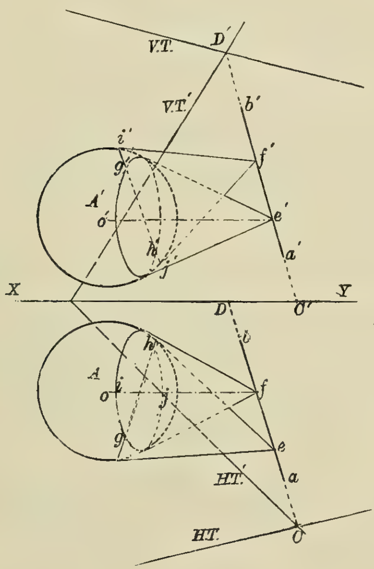


Fig. 192.

with their vertices on the given line, the intersection of the bases of these cones on the sphere will be points in the T. planes.

The projection of these cones will necessitate the drawing of the ellipses forming their bases; but it is possible to select positions by which we can dispense with two of the ellipses, *i.e.* by making one axis in each projection parallel to one of the co-ordinate planes.

Draw a line from *o*, the centre of the plan of the sphere,

parallel to XY, and meeting the plan of the given line in f. Find the point f' by a projector from f.

Draw lines from f' tangential to the sphere, meeting it in the points i' and j'. Join $i'j'$. This is the base of one cone. Draw projectors from this cone and complete its plan.

From o' draw a line parallel to XY, and meeting the elevation of the given line in e'. This is the axis of the second cone. Complete the plan and elevation as just described.

Having completed the projections of these cones, we find that their bases intersect each other on the sphere in the points gh and $g'h'$. g, g', are the projections of the point where the upper T.P. touches the sphere, and h, h', those of the lower T.P.

We have now determined the projections of three points in each T.P., viz. C Dg, C′D′g', of one plane, and C Dh, C′D′h', of the other.

From three given points in a plane we can determine its traces (Prob. 115). Draw the traces required. H.T., V.T., are the traces of the upper T.P., *i.e.* the one passing through the point G; and H.T.′, V.T.′, are those of the plane passing through point H. The projections of the points of contact are, of course, gg', and hh'.

It is interesting to know that if any number of cones were placed with their vertices on AB, and tangential to the sphere, their bases would intersect each other in the same point on the sphere.

Problem 159.

To determine the traces and the projections of the points of contact of a T.P. *to three given unequal spheres.* Fig. 193.

There are eight possible variations to these conditions, *i.e.* eight T. planes could be projected in accordance with the data furnished. These are illustrated in Fig. 194. We will take those shown in D and E.

Let A, B, C; A′, B′, C′, be the projections of the three given spheres.

If we assume two enveloping cones, tangential to each pair of spheres, the intersection of their bases will be points in the T. planes required.

Draw tangents to A and B meeting in e; also to A′ and B′ meeting in e'.

We must now make a supplementary elevation showing the full length of this cone.

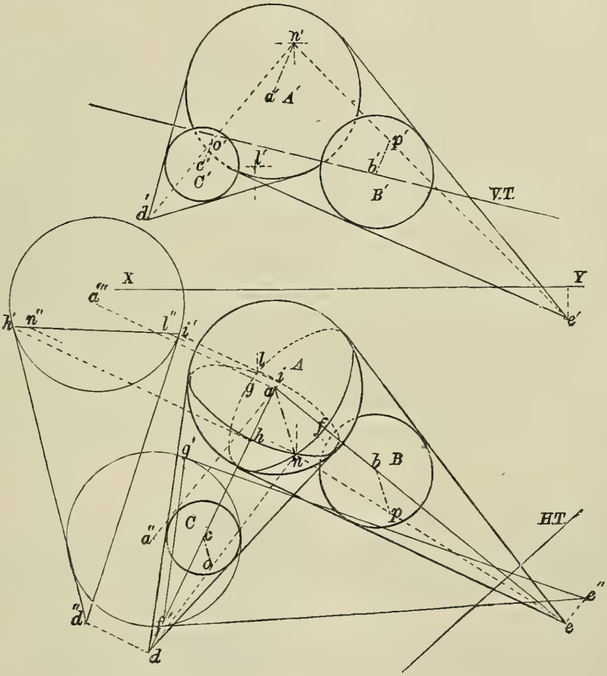


Fig. 193.

Draw a line from a, the centre of plan A, to e, and assume this to be a new ground line. Draw the projectors aa'', ee'', and make the length of aa'' equal to the height of a' above XY, also the length of ee'' equal to the height of e' above XY.

With a'' as centre, draw a circle equal to A. Draw lines, tangential to this circle, to e'', meeting it in the points f' and g'. Join $f'g'$. Then $e''f'g'$ is the supplementary elevation of the

cone tangential to sphere A. Project the base of this cone on A by means of projectors drawn from f' and g' perpendicular to ae and meeting it in f and g. Complete the ellipse forming the projection of the base of this cone. We will now project a cone enveloping the spheres A and C. Draw tangents, as before, and projections ad, $a'd'$, of its axis.

We shall require another supplementary elevation showing the side of this cone.

Assume the line ad to be another ground line. Draw the projectors aa''', dd'', respectively, equal to the distance of a' and d' above XY. With a''' as centre, draw another elevation of sphere A. From d'' draw the lines $d''h'$, $d''i'$, tangential to this circle. Join $h'i'$. Then $d''h'i'$ is the supplementary elevation of the second cone. Project its base by drawing projectors from h' and i' perpendicular to ad and meeting it in h and i. Complete the ellipse forming the projection of the base of this cone.

The bases of these two cones intersect each other in the points l and n on the sphere A.

To find these points in the supplementary elevation. Draw projectors from l and n perpendicular to ad, till they meet the

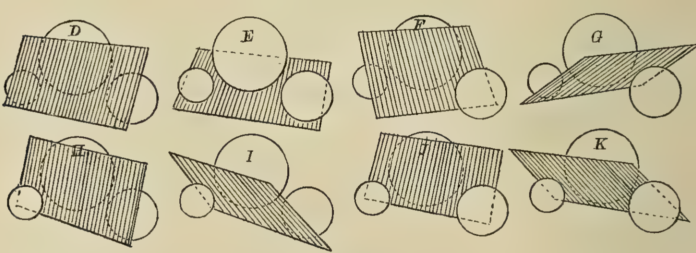


Fig. 194.

line $h'i'$ in the points l'' and n''. Take the distances these points are from the line ad, and measure them off on projectors from l and n above XY. These give the points l', n'.

We have now obtained the projections of three points in each of the T. planes illustrated in D and E (Fig. 194); viz.

den, *d′e′n′*, for T. plane shown in D, and *del*, *d′e′l′*, for the T. plane shown in E.

Find the traces of these planes (Prob. 115).

The H.T. and V.T. shown in Fig. 193 belong to the T. plane containing the points D, E, L. The traces of the T. plane containing the points D, E, N, come beyond the limits of the diagram.

To find where the T. plane that touches sphere A in the point N will touch the other spheres. Join *n* to *e*, and *n′* to *e′*; also *n* to *d*, and *n′* to *d′*. Join *an* and *a′n′*. From the centres *b* and *b′* draw lines parallel to *an* and *a′n′* till they meet the lines *ne* and *n′e′* in the points *p* and *p′*; also from *c* and *c′*, parallel to *an* and *a′n′*, till they meet the lines *nd* and *n′d′* in the points *o* and *o′*.

The points of contact for the other T.P. can be found in the same manner.

Problems on *Tangent Planes and Solids* constructed by *Horizontal Projection* will be found in the next chapter, from Prob. 185 to 194.

CHAPTER XXXII

HORIZONTAL PROJECTION

In *orthographic projection* (Chap. XXIII.) two planes are used perpendicular to each other, and the position of a point is ascertained by its projection on each of these planes. It is stated on page 204, "that the length of a projector determines the distance of a point from its projection." We also know that a projector is perpendicular to the plane on which the point is projected. If we were to mark the projection of a point on a plane by a number denoting the length of its projector, *i.e.* its distance from the plane on which it is projected, we should be able to ascertain its position from one projection only.

In *horizontal projection* the H.P. only is used, and figures are given indicating the perpendicular distances of the points either above or below the H.P. These figures are called "Indices." If the point is below the H.P., the negative sign ($-$) is placed before the figure. The unit which is generally used is one-tenth of an inch ($0{\cdot}1''$).

If the position of points can be thus determined, it necessarily follows that the inclination and length of lines, planes, etc., can also be found.

By *horizontal projection* many of the more complex problems in projection are very much simplified. It is of great utility in the drawing of forts, etc., as well as showing the levels and contours of surveys.

THE PLANE OF REFERENCE.

If the index numbers were high, it would be very inconvenient to have to measure them from the H.P., so to avoid this a horizontal plane can be taken at any assumed level, called "*the plane of reference.*" We can have any number of them to suit our convenience, either above or below the H.P., but we must make the necessary additions or deductions from the indices to suit the assumed level.

Fig. 195 is a perspective view in which H.P. is the horizontal

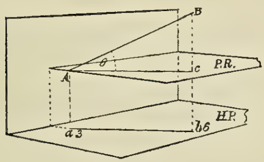


Fig. 195.

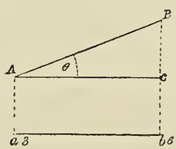


Fig. 196.

plane, which is always at the level of zero. P.R. is the plane of reference, and *ab* is an indexed line.

Fig. 196 shows the same line *ab* with its indices.

To find the true length and inclination of this line, assume *ab* to be a ground line, and erect perpendiculars at each end of it, *e.g.* make aA $= 0{\cdot}3''$, and bB $= 0{\cdot}6''$, and join AB. Then AB is the true length of the line. Instead of setting up the heights from the level of *a*, the H.P., we could draw a H. line at the level of A, as A*c*, and assume this to be the level of the plane of reference. As this level is $0{\cdot}3''$ above the H.P., we must deduct this height from the index number of B ($0{\cdot}6'' - 0{\cdot}3'' = 0{\cdot}3''$). *c*B will then be $0{\cdot}3''$ high. BA*c* gives θ, the angle AB forms with the H.P. Compare this with the perspective sketch (Fig. 195).

If the plane of reference is taken at the level of 15 and a point is indexed 25 (the unit being $0{\cdot}1''$ as before), the point will be 10 units or one inch above the plane of reference ($25 - 15 = 10$); or if a point on the same plane is indexed 10, it will be 5 units or $\frac{1}{2}$ an inch below the plane of reference ($10 - 15 = -5$).

Unless stated to the contrary, the plane of reference should be taken at the zero level, *i.e.* the level of the H.P.

If any point in a line bears the same index number as the plane of reference, that point is the H.T. of the line on the plane, *e.g.* A (Fig. 195) is the H.T. of AB.

THE SCALE OF SLOPE.

In horizontal projection, lines are indicated by their plan, which gives their direction on the H.P., their indices giving their inclination to this plane.

Planes are indicated by a plain scale called "*the scale of slope.*" This scale is always drawn perpendicular to the H.T. of the plane.

We know that a line contained by a plane, and perpendicular to the trace of the plane, always gives the inclination of the plane. Let us refer back to Fig. 107, in which the line *ab* is perpendicular to the line *ca*,—the H.T. of the plane A,—and gives the inclination of A to the H.P.

Let *ab* and *cd* (Fig. 197) be two horizontal lines. As *ab* is indexed zero, it must be on the H.P., and represents the H.T. of a plane containing the line *cd*. As *cd* is indexed 5, it must be 5 units above the H.P.

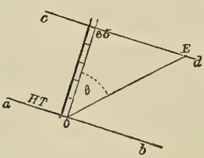


Fig. 197.

In any convenient position draw a plain scale perpendicular to these two lines, and divide it into as many equal parts as there are units between the level of the two lines (5). This will form a scale of slope. When this scale is made the plane is said to be determined.

Note.—The thicker line of the scale should always be on the left hand on ascending the slope; by this means we can always see which way the plane is inclined.

The scale of slope is generally used as a ground line upon which the elevation of the plane is constructed. *E.g.* to determine the actual slope of the plane containing the lines *ab*, *cd*, set off

on *cd* from the scale, $e\text{E} = 0{\cdot}5''$ in length—the number of units *cd* is above *ab*. Join E to the lower end of the scale. Then E$\text{O}e$ is θ, or the inclination of the plane to the H.P.

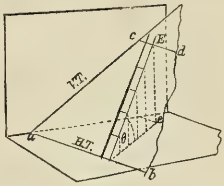


Fig. 198.

Fig. 198 is a perspective sketch illustrating this problem.

Note.—The scale of slope in the following problems is always indicated by the letter Z.

PROBLEM 160.

To draw a line from a given point parallel to a given line. Fig. 199.

Let *ab* be the given line, and *c* the given point. Draw *cd* parallel to *ab*, and equal to it in length.

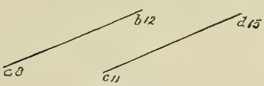


Fig. 199.

If we make the difference between the indices of *cd* equal to the difference between those of *a* and *b*, the lines will be parallel; *e.g.* if we make $c = 11$, then d must $= 15$, because *d* must exceed *b* as much as *c* exceeds *a*.

Note.—Lines parallel to each other on plan, and having the same difference in level for equal lengths, must be parallel to each other.

PROBLEM 161.

From a given indexed line ab, to determine its true length, H.T., *and* θ. Fig. 200.

Draw the perpendicular $a\text{A} = 0{\cdot}2''$ and $b\text{B} = 0{\cdot}5''$. Join AB, which is the true length of *ab*.

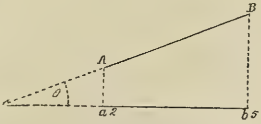


Fig. 200.

Produce AB till it meets *ab* produced. This gives the H.T. The inclination of AB to *ab* is θ.

Problem 162.

1. *From a given indexed line ab, to determine its true length,* H.T., *and θ.*
2. *To find the position of a point c2.*
3. *To determine the index of a given point d.* Fig. 201.

1. As a is prefixed by the negative sign (—), the perpendicular aA must be drawn below ab. Draw the perpendicular $b\text{B} = 0{\cdot}4''$. Join AB, which is the true length of ab.

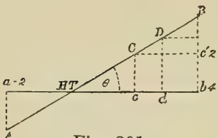


Fig. 201.

Where AB intersects ab is the H.T. The angle between the two lines is θ.

2. Set off on bB, $bc' = 0{\cdot}2''$. Draw c'C parallel to ab. Draw Cc perpendicular to ab. Then c is the position of the point required on ab, and C its position on AB.

3. Erect a perpendicular on d till it meets AB in D. The length of dD is the index required, which is 3.

Problem 163.

1. *From a given scale of slope* Z (*showing units*), *to determine the inclination of the plane to the* H.P.
2. *Through a given point* c′6 *to draw a plane parallel to* Z. Fig. 202.

It will be seen that only one index is given to Z, viz. $a12$. We know that this is the higher end of the scale, because the thick line is always on the left hand when ascending the slope; and as each division of the scale represents one unit, the index of b should be 7 $(12 - 5 = 7)$.

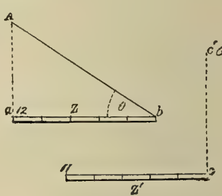


Fig. 202.

1. Assume the plane of reference to be at the level of b. Draw $a\text{A} = 0{\cdot}5''$ in length, perpendicular to ab. Join Ab. Then Aba is θ.

2. *Note.*—If planes are parallel, their traces are also parallel (Prob. 94); consequently, as the scale of slope is always perpendicular to the H.T. of the plane, the scales of slope of parallel planes must be parallel to each other.

Draw a line through the given point $c6$ perpendicular to Z. At any convenient point on this line draw another scale of slope Z', parallel to Z, and equal to it in length. As the index of b, the lower end of Z, is 7, and the lower end of Z' is 6, we must make the higher end of $Z' = 11$, *i.e.* one unit less than $a12$.

Problem 164.

Two parallel planes being given, to determine the distance between them. Fig. 203.

Let Z and Z' be the given scales of slope, and let the reference plane be at the level of 9.

Draw a perpendicular $bb' = 0{\cdot}3''$ $(12 - 9 = 3)$. Join ab'. Draw the perpendicular dd', and make $d' = 0{\cdot}4''$ above ab $(13 - 9 = 4)$.

Through d' draw a line parallel to ab'. In any convenient position draw $e'f'$ perpendicular to ab'. Then $e'f'$ is the distance between the parallel planes Z and Z'.

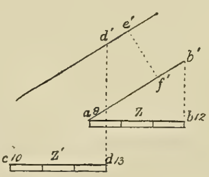


Fig. 203.

Problem 165.

Three points a, b, c being given, to determine the scale of slope of a plane containing them. Fig. 204.

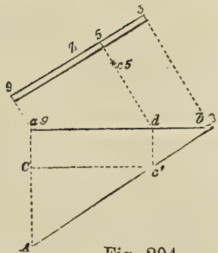


Fig. 204.

Join the two points that have the highest and lowest indices (a and b). Find the inclination of ab, as Ab (Prob. 161); the plane of reference to be at the level of 3. Determine the position of the point c on Ab (Prob. 162), as c'. Draw $c'd$ perpendicular to ab, and join cd. In any convenient

position draw the scale of slope Z perpendicular to *cd*, and draw lines from *a* and *b* parallel to *cd*. Figure the scale of slope with indices corresponding to *a* and *b*.

PROBLEM 166.

Two planes being given to determine the line of intersection between them—

1. *When the* H.T.s. *of the planes are parallel to each other.*
2. *When the* H.T.s. *are inclined to each other.*
3. *When the* H.T.s. *are nearly parallel.*

Figs. 205, 206, 207.

1. Let Z and Z′ be the two given planes (Fig. 205).

Take any line *xy* parallel to the scales of slope, and assume this to be the ground line of a plane of reference taken at the level of 9.

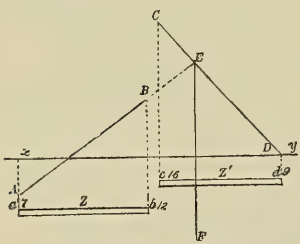


Fig. 205.

Draw perpendiculars at *c* and *d*. As *c* is indexed 16, we must make the point C 7 units above *xy* (16 $-$ 9 = 7). Join CD. This forms with *xy* the θ of Z′.

Draw perpendiculars at *a* and *b*. As *a* is indexed 7, we must draw A 2 units below *xy* (7 $-$ 9 = $-$ 2). The index of *b* being 12, make B 3 units above *xy* (12 $-$ 9 = 3).

Join AB, and produce it till it meets CD in E. This is where the two planes intersect. Draw EF at right angles to *xy*. This is the line of intersection required.

Note.—Either scale of slope could be taken as a ground line if desired. The line *xy* is taken to simplify the problem.

2. Let Z and Z′ be the two given planes (Fig. 206).

Take the plane of reference at the level 6.

Draw a perpendicular at *c*, and make *c*C 7 units in length (13 $-$ 6 = 7). Join C*d*, which makes with *cd* the θ of Z′

Find the position on C*d* of A′ and B′, their indices corre-

sponding to a and b of Z (Prob. 162). Draw lines from these points at right angles to Z' till they meet perpendiculars drawn from the points a and b, perpendicular to Z, in the points e and f. Draw a line through e and f, which is the line of intersection required.

Draw the line eE perpendicular to ef, 4 units in length.

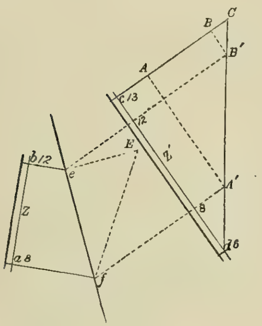


Fig. 206.

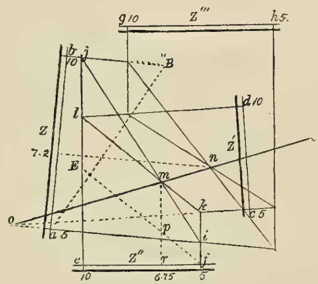


Fig. 207.

Join Ef. Then Efe is θ of the line of intersection between Z and Z'.

3. Let Z and Z' be the two given planes (Fig. 207).

We know from the preceding problem that the intersection of lines drawn from corresponding indices and perpendicular to each scale of slope will give points in the required line of intersection; but if these scales are nearly parallel, these points of intersection become so remote, sometimes, as to be inaccessible. When this is the case, other scales of slope are assumed, and lines are drawn perpendicular to them from corresponding indices. These lines will intersect those from the given planes in the same points.

Assume a plane Z'' with indices corresponding with those of Z and Z', and take the plane of reference at 5.

Note.—The indices are taken the same in each plane to facilitate the work; if they are of different values, corresponding indices could be found in the same way as shown in the preceding problem.

Draw perpendiculars to corresponding indices in each of the three scales of slope till they intersect: *e.g.* the perpendicular from $5Z''$ will meet that from Z in i, and the one from Z' in k; the perpendicular from $10Z''$ will meet that from Z in j, and the one from Z' in l.

Join ij and kl, intersecting each other in m. This is a point in the line of intersection required.

Assume another plane Z''', and proceed in a similar manner to find point n. Draw a line through m and n, which is the line of intersection required.

It will be seen that this line joins the point where 5Z intersects $5Z'$ in o. The other end of the line, if produced, would intersect the junction of 10Z with $10Z'$.

To index the points m and n. Find the inclination of plane Z'', as Ef (Prob. 163). Draw a line from m perpendicular to Z'', which gives pr, the required index, viz. 6·75. Find the index of point n in the same manner from Z. This is 7·2.

Note.—These indices could be found from any of the scales that are conveniently placed.

Problem 167.

From two given lines that intersect each other, to determine the angle between them. Fig. 208.

Let ab and bc be the two given lines.

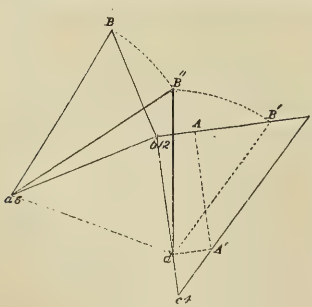


Fig. 208.

Find the inclination of ab, as aB; also the elevation of bc (Prob. 161).

Find on bc a point corresponding to the index of a, as d (Prob. 162).

As a and d have the same index value (6), we will take the plane of reference at that level.

Draw bB perpendicular to ab, and bB′ perpendicular to bc, each 6 units in length $(12 - 6 = 6)$.

With a as centre, and radius aB, draw an arc. With d as centre, and radius dB′, draw another arc till it meets the arc from B in B″. Join B″ to a and d. Then aB″d is the angle required. We have obtained the true lengths of three sides of a triangle abd; aB being the true length of ab, and B′d of bd. The intersection of these sides at B″ must be the angle between ab and bd.

Problem 168.

To draw a line perpendicular to a given plane through a given point. Fig. 209.

Let Z be the given plane and a the given point.

Use Z as the ground line, working from the level of 3.

Find the inclination of the plane, as fC (Prob. 163).

Determine the elevation of a, as a' (Prob. 161).

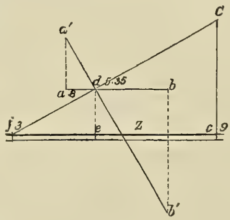


Fig. 209.

From a' draw $a'b'$ an indefinite length, at right angles to fC. Draw a projector from b' till it meets a horizontal line drawn from a in b. Then ab, $a'b'$, are the projections of the line required.

The index of the point of intersection d is equal to de (5·35).

Problem 169.

To draw a plane through a given point perpendicular to a given line. Fig. 210.

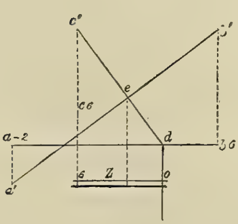


Fig. 210.

Let ab be the given line, and c the given point.

Find the elevation of the given line (Prob. 162), working from the level of the H.P., *i.e.* zero, and using ab as the ground line.

Find the elevation of c, as c', by

drawing a line through c perpendicular to ab, 6 units in length.

From c' draw a line at right angles to $a'b'$ till it meets ab in d, and intersects $a'b'$ in e.

Draw lines from d and e perpendicular to ab, and produce $c'c$. Draw the scale of slope Z parallel to ab, and index it with the levels of d and c', *i.e.* 0 and 6. This is the plane required.

Problem 170.

A plane and a right line being given, to determine their intersection. Fig. 211.

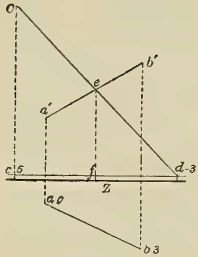


Fig. 211.

Let Z be the given plane, and ab the given line.

Use Z as the ground line, working from the level of d.

Find the inclination of the given plane, as dC (Prob. 163), and the elevation of the given line, as $a'b'$ (Prob. 161).

$a'b'$ intersects dC at e, which is the point required. The index of the point of intersection is equal to ef (1·5).

Problem 171.

To project a line from a given point, making an angle θ with the H.P., *and contained by a given plane.* Fig. 212.

This is similar to Prob. 109.

Let Z be the given plane and a the given point. Work from the level 4, using Z as the ground line.

Find the elevation of a by drawing a projector from a, 7 units above Z, as a' ($11 - 4 = 7$). From a' draw a line making the angle θ with the scale of slope, as $a'd$. This line is the generatrix of a cone.

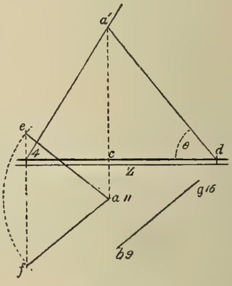


Fig. 212.

Draw a line through the level 4 perpendicular to Z. With *a* as centre, and radius equal to *cd*, draw an arc cutting the perpendicular through 4 in the points *e* and *f*. This arc is part of the base of a cone, of which *a′d* is the generatrix. Join *ea* and *fa*, either of which represents the plan of the line required. Join 4*a′*. This is the elevation of both *ea* and *fa*.

Problem 172.

To project a line inclined θ from a given point, and parallel to a given plane. Fig. 212.

Let Z be the given plane, and let the given point be *b*9.

Draw *bg* parallel to *fa*, and equal to it in length. As *a* is 7 units higher than *f*, *g* must be 7 units higher than *b*.

Problem 173.

To determine the scale of slope of a plane inclined θ and containing a given line. Fig. 213.

This is similar to Prob. 114.

Let *ab* be the given line, which is used as the ground line, working from the level of 5.

Find the elevation of *ab*, as *ab′* (Prob. 161).

From *b′* draw *b′c*, making the angle θ with *ab*. With *b* as centre, and radius *bc*, draw a circle. This is the base of a cone of which *b′c* is the generatrix. Any plane tangential to this cone, and containing the given line, must be the plane required.

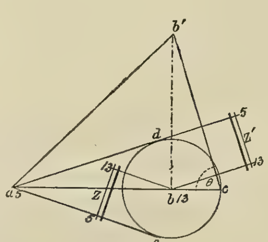


Fig. 213.

Draw *ae* tangential to the circle, and a line from *b* parallel to it. In any convenient position draw Z perpendicular to these

two lines, and index it with the levels of *a* and *b*. This is the scale of slope required.

There are two planes that would fulfil the required conditions: the second is found by drawing the tangent *ad*, and a line from *b* parallel to it. Draw the scale Z′ as described for Z.

Problem 174.

To determine a plane making a given angle with a given plane and intersecting it in a given line. Fig. 214.

This is similar to Prob. 100.

Let Z be the given plane, C the given angle, and *ab* the given line.

Use Z as the ground line, working to the level of 6.

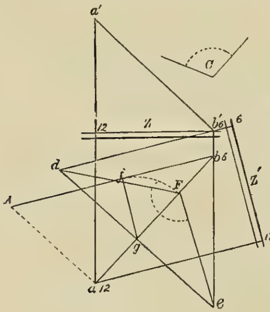


Fig. 214.

Make an elevation of *ab*, as *a′b′* (Prob. 161).

Assume any line *de*, drawn at right angles to *ab*, and cutting it in *g*, as the H.T. of a plane perpendicular to the line *ab*.

Make an elevation of the line *ab*, using *ab* as a plane of reference, as A*b* (Prob. 161). Draw *gf* perpendicular to A*b*. With *g* as centre, "construct" *f* on to the plane of reference *ab* in F. Draw a horizontal line in plane Z at the level of *b*, as *b′e*. Join F*e*. Draw F*d*, making an angle with F*e* equal to the given angle C, and meeting *de* in *d*.

As *de* is the H.T. of a plane, it must be on the plane of reference, which is at the level of 6; so *d* is at the same level as *b′*. Draw a line through *db′*, and make Z′ perpendicular to it. Draw a line from *a* parallel to *db′*. Index Z′ with the levels of *b′* and *a*. This is the plane required.

Problem 175.

To determine the angle between two given planes. Fig. 215.

This is similar to Prob. 99.

Let Z and Z′ be the given planes, and 7 the level of the plane of reference.

Note.—These planes have the same indices to simplify the construction. If they were of different values, points in each scale could be found to correspond by Prob. 166, 2.

Draw lines from the corresponding levels perpendicular to each scale of slope, till they meet in the points *a* and *b*.

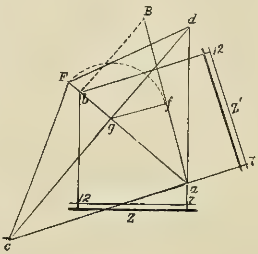


Fig. 215.

Draw any line *cd* at right angles to *ab*, cutting it in *g*, and produce it till it meets lines drawn through *a* produced; *cd* will then be at the level of the plane of reference, and represent the H.T. of a plane perpendicular to *ab*.

Make an elevation of *ab*, using *ab* as a ground line, as *a*B (Prob. 161).

Draw *gf* perpendicular to B*a*. With *g* as centre, construct *f* on to the plane of reference, as F. Join F*c* and F*d*. Then *c*F*d* is the angle required.

Problem 176.

To determine the angle a given line ab will form with a given plane Z. Fig. 216.

This problem is similar to Prob. 106.

The principle of this problem is explained at the commencement of Chap. XXVI.

Draw the inclination of the given plane, as *c*D, working from the level 6 (Prob. 163).

Find the elevation of the given line, as *a′b′* (Prob. 161).

From *a′* draw *a′e′* perpendicular to *c*D. Draw the projector *e′e* till it meets a line from *a* parallel to Z. *ae* is the plan of *a′e′*.

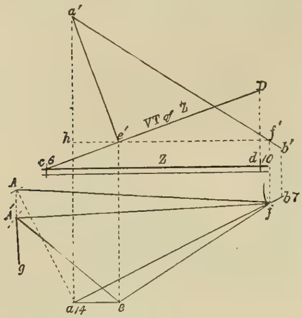


Fig. 216.

Draw a line from *e′* parallel to Z till it meets *a′b′* in *f′*. Draw the projector *f′f* till it meets *ab* in *f*. Join *ef*. This is the plan and true length of the line EF.

To determine the true length of the line AF, draw the line *a*A perpendicular to *af*, and equal in length to *ha′*. Join A*f*.

We now have the true lengths of the three lines forming the triangle, whose projections are *aef*, *a′e′f′*. To find the angle between *ae* and *af*, we must construct this triangle on to an H. plane at the level of *ef*. With *e* as centre, and radius equal to *e′a′*, draw an arc. With *f* as centre, and radius *f*A, draw another arc, cutting the one drawn from *e* in A′. Join A′*e* and A′*f*.

Then *e*A′*f* is the angle between the given line and a perpendicular to the given plane. Find the complement of this angle by making A′*g* perpendicular to A′*f*. Then *g*A′*e* is the angle required.

Problem 177.

To determine the scale of slope of a plane perpendicular to a given plane and drawn through a given line. Fig. 217.

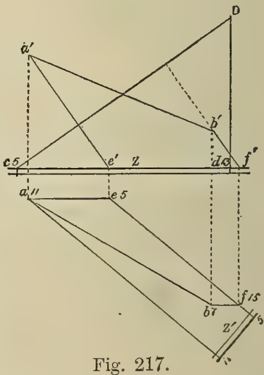


Fig. 217.

Let Z be the given plane and *ab* the given line.

Find the inclination of Z, working from the level of 5, as *c*D (Prob. 163).

Draw lines from *a′* and *b′*, perpen-

dicular to cD, till they meet the plane of reference in the points e' and f'. Draw projectors from these points till they meet lines drawn from a and b, parallel to Z, in the points e and f. Join ef. This line is the H.T. of a plane perpendicular to Z. At any convenient point on ef produced, draw Z′ perpendicular to it, and from a draw a line parallel to ef. Index Z′ with numbers corresponding to a and e; it will then be the scale of slope required.

Problem 178.

Through a given point e to draw a line to meet two given lines ab and cd, inclined to each other and not in the same plane. Fig. 218.

Find the elevations of ab and cd, as aB and cD (Prob. 161).

Find a point on each of the given lines of the same index as the given point e (10), as g and h (Prob. 162).

Draw a line from g through e, and make the scale Z perpendicular to it; also draw a line from e through h, and make the scale Z′ perpendicular to it.

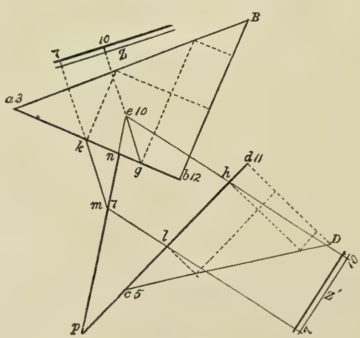


Fig. 218.

Find another point on each of the given lines bearing the same index, 7 for example, as k and l (Prob. 162).

Draw a line from k parallel to eg, and from l parallel to eh, till they meet in m. Draw a line from e through m intersecting ab in n, and cd in p. Then ep is the line required, *i.e.* it is the intersection of two planes, each of which contains one of the given lines and the given point, the given point being contained by the line of intersection also.

Problem 179.

1. *To draw a plane through a given line ab parallel to a given line cd.*
2. *To draw a line perpendicular to two given lines inclined to each other and not in the same plane,* e.g. *the lines ab and cd.* Fig. 219.

1. From any point in the given line *ab* draw a line *be* parallel to *cd* (Prob. 160).

On *ab* determine a point *f* with the same index as *e* (Prob. 162).

Draw a line through *e* and *f*, and in any convenient position

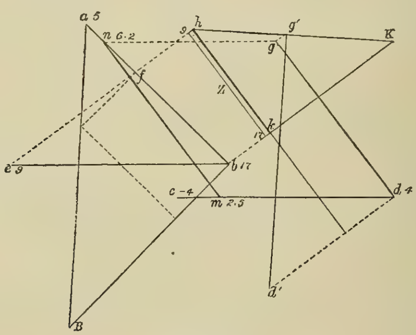


Fig. 219.

draw the scale Z perpendicular to it. Index the scale with figures corresponding to the levels of *b* and *e*. This is the plane required.

2. Find the inclination of Z as *h*K (Prob. 163). Determine the elevation of *d*, as *d′*, using Z as a ground line and working from the level of *h* (Prob. 161).

Draw a line from *d′* perpendicular to K*h* and meeting it in *g′*. Draw a line from *d* parallel to Z till it meets a projector from *g′* in *g*. Then *dg, d′g′*, are projections of a line perpendicular to the plane Z.

Draw a line parallel to *gd*, and equal to it in length, till it meets *ab* and *cd* in the points *n* and *m*.

This is the line required; its indices (n 6·2 and $m-2$·5) are determined from elevations of the lines ab and cd (Prob. 162).

PROBLEM 180.

To determine the scale of slope of a plane passing through a given point a on the surface of a sphere. Fig. 220.

Let c be the centre of the given sphere.

Join ca, and assume it to be a ground line. Make an elevation of a, as a', working from the level of c (Prob. 162).

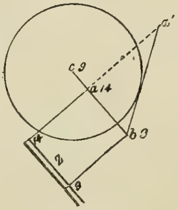


Fig. 220.

Assume the plan of the sphere to be its elevation, and from a' draw a line tangential to the elevation of the sphere till it meets ca produced in b. Draw lines from a and b perpendicular to cb, and draw Z perpendicular to them. Index Z to correspond with the points a and b. This is the scale of slope required.

PROBLEM 181.

From a given point a to determine a plane making an angle θ with the H.P., *and perpendicular to a given plane.* Fig. 221.

Let Z be the given plane.

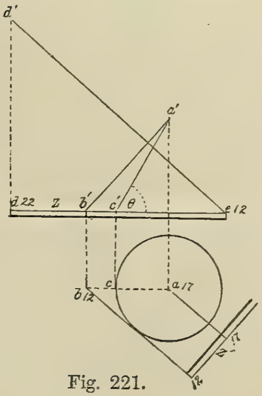


Fig. 221.

Find the elevation of Z (Prob. 163), also of the given point a, as a' (Prob. 162), working from the level of e. From a' draw a line making the angle θ with Z, and meeting it in c'. From c' draw a projector till it meets a line from a, parallel to Z, in c. With a as centre, and radius ac, draw a circle. This is the base of a cone of which $a'c'$ is the generatrix.

From a' draw a line perpendicular

to the elevation of Z ($d'e$), and meeting Z in b'. Draw a projector from b' till it meets ac produced in b.

From b draw a line tangential to the base of the cone. From a draw a line parallel to the tangent from b, and draw the scale of slope Z′ perpendicular to them. Index Z′ with numbers corresponding with the points a and b. This is the scale of slope of the plane required.

Problem 182.

To determine the T.P. *to a given cone, with its base resting on the* H.P.; *the* T.P. *to contain a line drawn from a given point* a, *and to be inclined to the* H.P. *at an angle of* 30°. Fig. 222.

Let c be the plan of the vertex of the given cone. Join ac, and let it represent xy.

Find a' and c', the elevations of the points a and c, (Prob. 162).

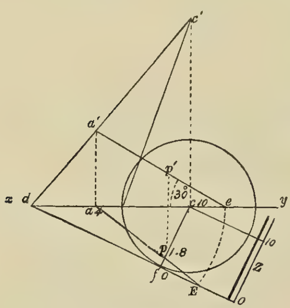


Fig. 222.

Join $c'a'$, and produce the line till it meets xy in d. This line contains the given point, and must be in a plane tangential to the cone, as it contains the vertex c'.

From d draw a tangent to the base of the cone, touching it in f. This is the H.T. of the plane required. Produce df, and draw a line from c parallel to it. Draw Z perpendicular to these lines, and index it to correspond with the points f and c. This is the scale of slope of the plane required.

To determine the plan of the line $a'e$ and its point of contact with the cone. Assume $a'e$ to be the generatrix of a cone. With a as centre, and radius ae, draw an arc till it meets the H.T. in E. This arc is a portion of the base of the cone, of which a' is the vertex. Join aE, which is the plan of a generatrix, and

must consequently be at the given angle (30°) with the H.P. Draw *fc* perpendicular to the H.T. This is the line of contact between the cone and the T.P., and where it intersects *a*E in the point *p* is the point of contact between the line and the cone.

To determine the index of *p*, find its elevation (p'). Then its height above XY is the index required.

Problem 183.

To determine a plane tangential to a given cone and sphere, the T.P. *to pass between them.* Fig. 223.

This problem is similar to Prob. 151.

Let *a* be the centre of the given sphere, and *b* the plan of the vertex of the given cone.

Draw a line through *ab* and assume it to be the level of the H.P.

Find the elevations of the points *a* and *b*, as a' and b' (Prob. 162).

Draw an elevation of the sphere (with a' as centre). Also draw the elevation of the cone.

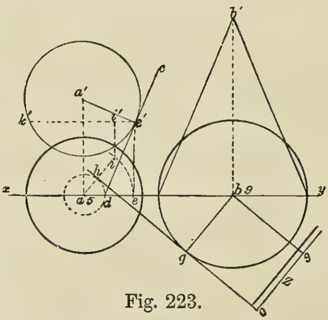


Fig. 223.

Draw a tangent *cd* to the elevation of the sphere (a'), parallel to the side of the cone, and touching it in e'. With *a* as centre, and radius *ad*, draw a circle. This is the H.T. of a cone enveloping the sphere a'. Draw a tangent to this circle and the base of the cone, passing between them, as *hg*. This is the H.T. of the T.P. required.

Produce *hg*, and draw a line from *b* parallel to it. Draw Z perpendicular to these lines, and index it to correspond with the points *h* and *b*. This is the scale of slope of the T.P.

Join *gb*. This is the line of contact with the cone.

From e' draw $e'k'$ parallel to *xy*. This is the circle of

contact between the sphere and the enveloping cone. Draw the projector $e'e$. With a as centre, and radius ae, draw an arc. Draw a line from a, perpendicular to hg, till it meets this arc in i. This is the point of contact.

To determine the index of i, find its elevation, as i'. The height of this point above xy is the index required.

Problem 184.

Through a given point a on the surface of an ellipsoid, to determine a plane tangential to it. Fig. 224.

Let *cedf* be the plan of the ellipsoid, and a the given point on its surface; assume the plane of reference to be at the level 9.

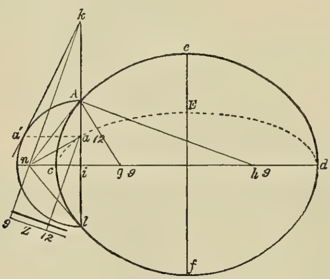


Fig. 224.

Draw a line kl through a, perpendicular to the longer axis of the solid, and cutting it in i and the plan of the solid in l. With i as centre, and radius il, draw a semicircle. This represents a half-section of the solid, taken on the line kl, and "constructed" on to the plane of reference.

Draw the line aa' perpendicular to kl.

If we assume kl to be the ground line of the plane of reference, a' is the elevation of a.

Note.—All lines tangential to a curved surface, and passing through the same point on that surface, must be contained by the same plane.

If we determine two lines tangential to the solid, and passing through point a, a plane containing these two lines will be the one required. Draw a tangent to a', meeting lk in k. Then k is the H.T. of the tangent; its plan is ak. Take a longitudinal section through the solid, passing through point a

and the axis, *e.g.* on the curve *ca*E*d*, and rotate it on the axis till it is on the plane of reference. It will then coincide with the curve *c*A*ed*. Draw a tangent to point A (Prob. 183, Plane Geometry), meeting the plane of reference in *n*. This is the H.T. of the tangent A*n*, and *an* is its plan. If we join *nk*, *i.e.* the H.T.s of the two tangents, we obtain the H.T. of a plane containing them. Produce *kn*, and draw a line from *a* parallel to it. Draw Z perpendicular to these two lines, and index it to correspond with the points *n* and *a*. This is the scale of slope of the plane required.

Problem 185.

Let two given lines ab and cd, inclined to each other and not in the same plane, represent the axes of two cylinders of equal diameters which touch one another. Project the plan of the cylinders, and determine the point of contact. Fig. 225.

To simplify this problem, let the two given lines represent

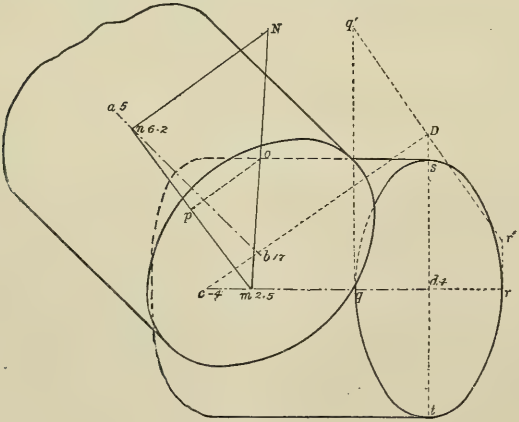


Fig. 225.

ab and *cd* in Prob. 179. We know that the point of contact must be on a line perpendicular to both axes: the line *nm* (Prob. 179) fulfils these conditions.

The lines *ab*, *cd*, and *nm*, Fig. 225, are precisely similar to the corresponding lines of Fig. 192, and are indexed with the same numbers.

The point of contact must be on the line *nm*; and as each cylinder is to have the same diameter, the point of contact will be on the centre of this line. Make an elevation of *nm* as follows. As *n* is $-2{\cdot}5$, draw *n*N 3·7 long perpendicular to *nm* ($6{\cdot}2 - 2{\cdot}5 = 3{\cdot}7$). Bisect *nm* in *o*. Then *m*O is the radius of each cylinder.

Make an elevation of the line *cd*, as *c*D (Prob. 161). Through D draw $q'r'$ perpendicular to *c*D. Set off the radius of the cylinder (*m*O) on each side of D in the points of q' and r'. Draw lines from q', r', perpendicular to *cd* produced, till they meet it in the points *q*, *r*. Set off on the perpendicular through *d* produced, *ds* and *dt*, each equal to *m*O. Then *st* and *qr* will be the major and minor axes of an ellipse which forms the end of one of the cylinders required, of which *cd* is the axis. Draw the ellipse (Prob. 181, Plane Geometry). Proceed in the same manner with the axis *ab*, and complete the other cylinder.

Draw a line from O perpendicular to *nm*, and meeting it in *p*. This is the point of contact between the two cylinders.

Problem 186.

To determine the T.P. *to a given cylinder inclined to the* H.P. *and parallel to a given line cd.* Fig. 226.

Let *ab* be the plan of the axis, and *e* the projection of a point in the circumference of the base of the cylinder.

Find the elevations of *a* and *e*, working from the level of *b*, as a' and e' (Prob. 162). Join $a'b$, which is the elevation of the axis. Also join be', which gives the radius of the base of the cylinder. With *b* as centre, and radius be', draw a semicircle meeting a perpendicular through *b* in the points *f* and *h*. Produce $e'b$ to g', making bg' equal to be'. Then $g'e'$ is the elevation of the base of the cylinder. Draw the projector $g'g$.

Then *fh* is the major, and *ge* the minor axis of the ellipse forming the projection of the base of the cylinder. Draw the ellipse (Prob. 181, Plane Geometry).

From *a* draw *ak* parallel to the given line *cd* (Prob. 160). Draw *k*K perpendicular to *ab*. From *a′* draw a line through K till it meets *e′g′* produced in *l′*. Find the plan of this line by producing *ak* till it meets a perpendicular to *ab* through *l′* in *l*. Join *lb*, which is a line of intersection of a plane containing *ab*

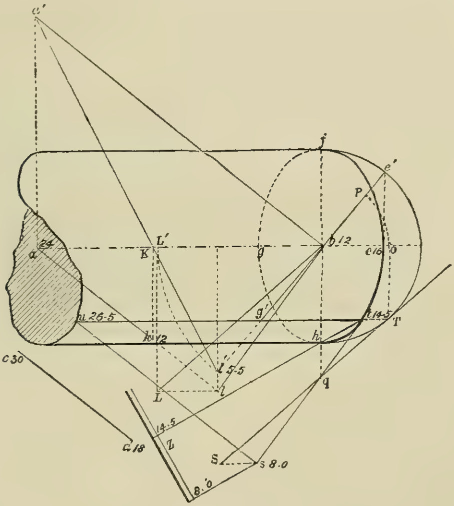


Fig. 226.

and a line *ak* (parallel to the given line), with the plane of the base. To construct this line on to the plane of reference. With centre *b*, and radius *bl′*, draw an arc till it meets *ab* in L′. Draw a projector from L′ till it meets a horizontal line from *l* in L. Join *b*L, which is the line *bl* "constructed" on to the plane of reference.

This line is the H.T. of a plane containing the axis and a line parallel to the given line *cd*, and the T.P. will be parallel to this plane.

The semicircle drawn through *e′* is a part of the base of the cylinder also "constructed" on to the plane of reference. Draw

a tangent to this semicircle, parallel to Lb, and cutting it in T. From T set off TS equal to bL in length. Also draw the line Tu parallel to ab, cutting the projection of the base of the cylinder in t. Make tu equal to ab in length. From u draw a line parallel to ak till it meets a line from S, parallel to ab, in s. Join st, which is parallel to lb, and represents the tangent ST "rotated" on the centre q till it is in the plane of the base of the cylinder.

The line tu should be indexed to represent the same inclination as ba. To determine the index of t. Draw the line To perpendicular to ab. With b as centre, and radius bo, draw an arc till it meets the elevation of the base in P. The height of P above ab produced is 2·5. This added to the index of b gives 14·5 for t. The index of point l (5·5) is determined from its elevation (l'), which is 6·5 below the plane of reference 12 ($12 - 6{\cdot}5 = 5{\cdot}5$). The line us is indexed with the same inclination as al.

The scale of slope (Z) is determined by Prob. 165, the three given points being s, t, and u. Index the scale from the points s and t.

Problem 187.

To determine the T.P. *to a given cone with its base inclined to the* H.P.; *the* T.P. *to pass through a given point p.* Fig. 227.

Let ab be the plan of the axis, and c the projection of a point in the circumference of the base.

Find the elevations of the points b and c, as b' and c' (Prob. 162), working from the level of a. Join ab', which is the elevation of the axis ab. Also join ac', which is the elevation of half of the base. A line joining $c'b'$ is a generatrix of the cone. With a as centre, and radius ac', draw a semicircle till it meets a perpendicular through a in fg; this is half of the base of the cone "constructed" on to the plane of reference. From a on ab set off ae equal to ac. Then fg is the major and ce the

minor axis of the ellipse forming the plan of the base. Draw the ellipse (Prob. 181, Plane Geometry), and complete the projection of the cone.

Find the elevation of p, as p'. Draw a line from b' through p' till it meets the elevation of the base produced in h'. Also draw a line from b through p till it meets a projector from h' in h.

Construct the point h' into the plane of reference, as H′.

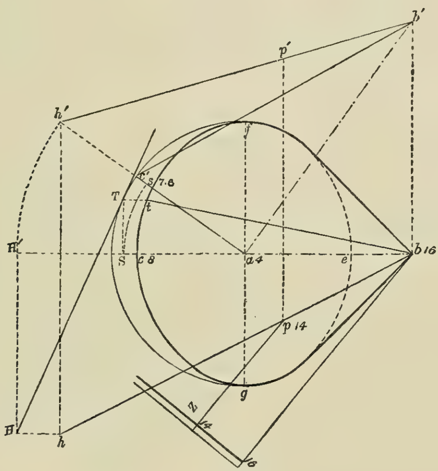


Fig. 227.

Draw a projector from H′ till it meets a horizontal line from h in H. Then H is the H. trace of a line containing the vertex B and the given point P.

From H draw a tangent to the semicircle, touching it in T. Draw a line from T, parallel to ab, till it meets the projection of the base in t. Join tb, which is the line of contact.

To determine the index of t. Draw TS perpendicular to ab. With a as centre, and radius aS, draw an arc till it meets the elevation of the base in s. The perpendicular height of s above ab (3·6) added to the index of a (4) gives its index (7·6).

The scale of slope (Z) is determined by Prob. 165, the three given points being t, b, and p. Index the scale from the points p and b.

Problem 188.

The axis of a cone with its base inclined to the H.P. *being given, as well as the centre of a sphere, to determine the projections of both solids, with their point of contact.* Fig. 228.

Let *ab* be the given axis and *c* the given centre. Join *ac*, and on this inclined line find a level corresponding to the index *b*, as B (Prob. 162, 2). With *c* as centre, and radius *c*B, draw

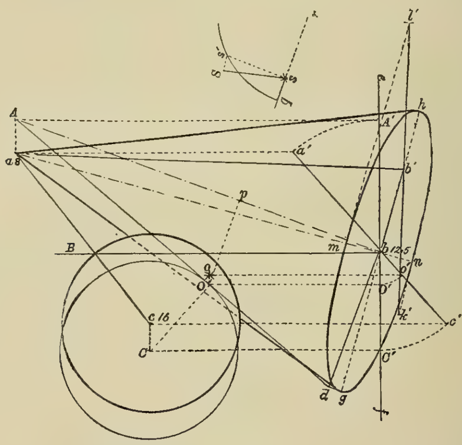


Fig. 228.

the plan of the sphere. Join B*b*, and through *b* draw *ef* perpendicular to it.

Assume *ef* to be a ground line, and find the elevations of the points *a* and *c*, working from the level of *b*, as *a′c′* (Prob. 162). Join *a′c′*. Construct these points on to the plane of reference, as A′C′, and find their horizontal projections, as AC. With C as centre, draw an auxiliary plan of the sphere. Join A*b*. From A draw a line tangential to the auxiliary sphere till it meets a perpendicular to A*b* in *d*, and touching the auxiliary circle in O.

Draw *gh* perpendicular to *ab*, and set off along it, on each side of the point *b*, a distance equal to *bd* in the points *g* and *h*.

Then *gh* is the major axis of the ellipse forming the projection of the base.

Now assume *ab* to be a ground line, and find the elevation of the point *b*, working from the level of *a*, as *b′*. Join *b′a*, which is the elevation of the axis. Draw the line *k′l′* perpendicular to *ab′*, and set off along it, on each side of the point *b′*, a distance equal to *bd*, in the points *l′* and *k′*. Then *l′k′* is the V. trace of the plane of the base. From *l′* and *k′* draw lines perpendicular to *ab* till they meet it in the points *m* and *n*. Then *mn* is the minor axis of the ellipse forming the projection of the base. Draw the ellipse (Prob. 181, Plane Geometry).

To determine the point of contact. Draw O*O′* perpendicular to *ef*. With *b* as centre, and radius *b*O′, draw an arc till it meets *a′c′* in *o′*. Draw O*p* perpendicular to A*b*. Draw a projector from *o′* till it meets O*p* in *o*, which is the point of contact required.

To determine the index of the point of contact *o*, we will make a supplementary drawing, to avoid confusion in the diagram.

With *r* as centre, and radius equal to O*p*, draw an arc till it meets a line drawn through *r* in *q*. Assume *qr* to be a ground line at the level of *p* (10·7). The index of *p* is determined from *ab′*. Set off *rs* equal to the distance *o* is from the axis *ab*, and draw *ss′* perpendicular to *qr*. Then *s′* is the elevation of the point of contact on a V. plane. We must now "rotate" it till it is in a plane perpendicular to the inclined axis *ab′*. Make the angle S*sq* equal to the angle the plane of the base forms with the plane of reference. With *s* as centre, and radius *ss′*, draw an arc till it meets the line S*s* in S. The perpendicular height of S above *qr* (3) + 10·7 = 13·7.

Problem 189.

The axes of a cylinder and cone, with the diameter of the cylinder, being given, to determine the projection of the cone, and its point of contact with the cylinder. Fig. 229.

Let *ab* be the axis of the cylinder, and its diameter 6 units; also let *cd* be the axis of the cone, and *d* its vertex.

Find the elevation of *ab*, as *a′b*, working from the level of *b*. Through *b* draw the line *no* perpendicular to *ab*, and *hk* perpendicular to *a′b*, and set off on these lines from *b*—*h*, *n*, *k*, and *o*—equal to the given radius (3). Draw lines from *h* and *k* perpendicular to *ab* till they meet it in the points *l* and *m*. Then *no* is the major, and *lm* the minor axis of an ellipse forming the projection of the base of the cylinder. Draw the ellipse (Prob. 181, Plane Geometry). Also draw lines from *n* and *o* parallel to *ab*, to complete the cylinder.

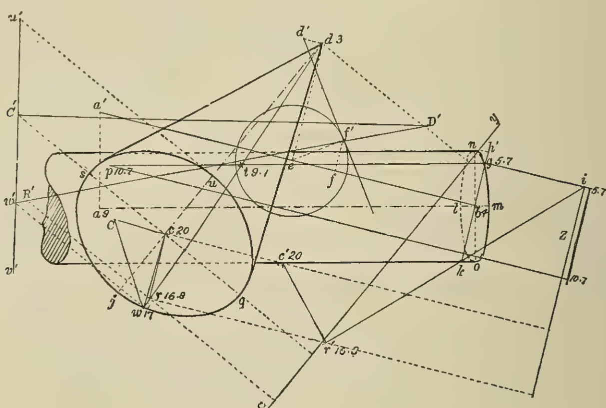


Fig. 229.

Determine a tangent plane to the cylinder, containing the vertex of the cone, as follows :—

Find the elevation *d′* of the vertex *d*. Draw *d′e* perpendicular to the axis *a′b*. Assume *on* to be a new ground line. With *b* as centre, and radius *bo*, draw a semicircle. Draw a projector from *d* till it meets *on* produced in E. On this projector set off ED equal to *ed′*.

The semicircle represents a section of the upper part of the cylinder, and D the position of the vertex of the cone taken on a plane at *e*, perpendicular to the axis *a′b*.

From D draw a tangent touching the semicircle in G. Draw G*g* parallel to the axis *ab*.

The index of g is determined by taking the distance of G from on, setting it off from b on bh, and then finding its perpendicular height above ab $(2) + 4 = 6$.

From g draw gp parallel to ab, and equal to it in length, and indexed for the same inclination with the plane of reference (Prob. 160). This is the line of contact of the T.P. with the cylinder. To find its scale of slope (Z), draw lines from g and p parallel to $a'b$, and draw Z perpendicular to them. Figure the scale of slope with indices corresponding with the points g and p.

Note.—A plane containing the axis of the cone, and perpendicular to the T.P., will also contain the line of contact between the cone and the T.P. This line of contact is the line of intersection between the two planes.

Draw the elevation of Z, as ir' (Prob. 163), using Z as the ground line, and working from level of i (6·0). Find c', the elevation of c. As the point c is indexed 20, we must make its elevation (c') 14 above Z $(20 - 6 = 14)$.

From c' draw a line perpendicular to the elevation of the T.P. and meeting it in r'; its index is its distance from Z $(11) + 6 = 17$. Draw a line from c parallel to Z till it meets the projector $r'r$ in r. Join rd. This is the line of contact between the cone and the T.P. Where this line intersects the line of contact of the cylinder in t gives the point of contact between the two solids.

To determine the base of the cone, assume cd to be a ground line, and find the elevations of the points c, d, and r.

Note.—As the index number of c, if set off from cd, would occupy too much space, we can lower the plane of reference to suit our convenience; *e.g.* draw xy parallel to cd and 10 units below it, and measure the heights from this line; or we could deduct 10 units from each of the indices and measure them above cd.

The elevations of the three points required are C′, D′, and R′. Join C′D′. This is the elevation of the axis of the cone. Through C′ draw a line perpendicular to C′D′, as $u'v'$. The elevation of the plane of the base is on this line.

Join D′R′. This is the elevation of the line of contact of

the cone. Produce D′R′ till it meets $u'v'$ in w'. From w' draw a perpendicular to cd till it meets dr produced in w. Join wc. The index of w is determined by measuring the distance of w' from xy.

To determine the radius of the base of the cone, "construct" cw on to the plane of reference by making cC perpendicular to cw, and equal in height to the difference between the indices of c and w ($20 - 17{\cdot}7 = 2{\cdot}3$). Join C$w$, which is the radius required. With c as centre, set off on the line through cC′ the points s and q each equal to Cw. This is the major axis of the ellipse forming the projection of the base of the cone. From C′ set off C′u' equal to cw. Draw $u'u$ perpendicular to cd. From c set off on cd produced, cj equal to cu. Then ju is the minor axis of the ellipse. Draw the ellipse, and complete the cone.

The index of the point of contact t is obtained from the scale of slope (Z).

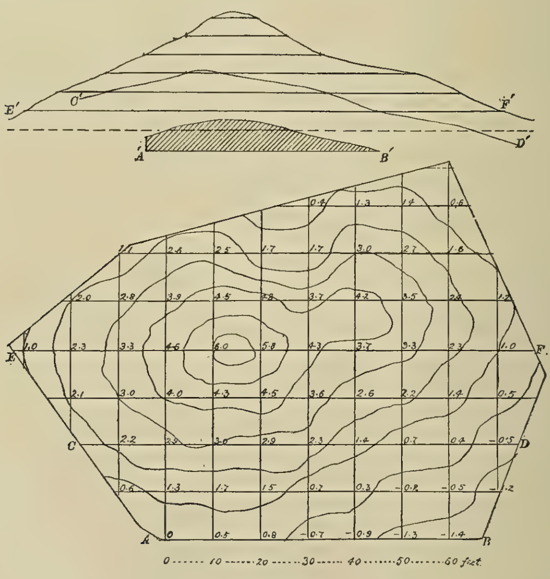


Fig. 230.

The application of horizontal projection to the plan of a piece of land is shown in Fig. 230.

The indices show the levels of the different points above or below the zero level, and are taken 10 feet apart. Each curved line, called a "*contour*," is at an even level throughout its course. The difference of level between each contour is one foot.

The upper part of the figure shows a section and two elevations of the plan. The section is taken on the line AB, and the elevations on the lines CD and EF. The vertical heights are usually drawn to an enlarged scale (in this instance four times that of the plan), so that the inequalities of the land can be more readily seen. The dotted line shows the zero level, and the horizontal lines represent the levels of the different contours.

In practice, it is usual to take a plane of reference below the lowest level, called the "*datum level*," so as to avoid using the minus sign.

Regular plane figures can be determined from their plans without their indices being given, when they are in a V.P., as illustrated by the two following problems.

Problem 190.

From the given plan of a square to determine its elevation. Fig. 231.

Let *abdc* be the horizontal projections of the corners of the given square.

From *a* draw a line perpendicular to *ac* and equal to it in length, as *ad'*. From *a* on *ad'* set off distances equal to *b* and *d*, as *c'* and *a'*. Draw projectors from *d* and *c* till they meet horizontal lines drawn from *d'* and *c'* in the points D and C. Join *ba'*, *b*C, CD, and D*a'*, which is the elevation of the square required.

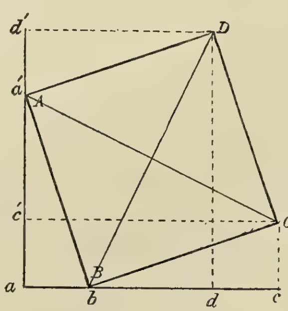


Fig. 231.

Note.—If two lines of equal length meet in the same point, their projections on two planes perpendicular to the plane containing the two lines, and forming the same angle with each other as the angle contained by the two lines, will have the following proportion:—

The sum of the projections of one line on both planes is equal to the sum of the projections of both lines in one plane: *e.g.* let ac and ad' (Fig. 231) be the edges of two planes at the same angle with each other as the two lines AB and AD (90°), and the dotted lines the projectors; then $ad + a'd' = aa' + a'd'$, or $dc + d'c' = dc + ad$.

The sum of the projections of each diagonal is also equal; *e.g.* $ac + a'c' = bd + ad'$.

Problem 191.

From the given plan, abc, of an equilateral triangle, to determine its elevation. Fig. 232.

First determine an auxiliary ground line (X′Y′), making the same angle with XY as the angle between two of the sides of the given triangle (60°).

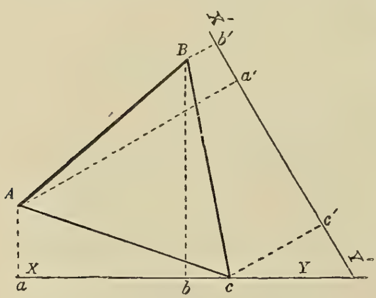


Fig. 232.

At any convenient point on XY draw X′Y′ at an angle of 60° with it. Draw the projector cc' perpendicular to X′Y′. From c' set off $c'b'$ equal to ca, and $c'a'$ equal to ba.

Draw projectors from a' and b' till they meet projectors drawn from a and b in the points A and B. Join AB, Bc, and cA, which is the elevation required.

CHAPTER XXXIII

PROJECTION OF SHADOWS

PART I.—SHADOWS USED IN PRACTICAL WORK

SHADOWS are of two kinds, viz. :—those cast from a source of light at an infinite distance, when the rays of light are practically parallel to each other, as is the case with the light from the sun; and those cast by a luminous point,—a lighted candle, for instance,—when the rays are divergent.

In all mechanical projections of shadows, unless specially stated to the contrary, the rays are supposed to be parallel to each other; then if the direction of the light is given, the projection of the shadow can be determined.

In all architectural and engineering drawings a conventional treatment of shadows is used; the light is assumed to be in a fixed direction, that of the diagonal of a cube, as the angle *a′ba*, Fig. 233. This direction has special advantages, viz. it makes equal angles with each co-ordinate plane, and its projections on the H.P. and V.P. are each 45° with XY. *ab* is the H. projection, and *a′b′* the V. projection of the line *a′b* (Fig. 233).

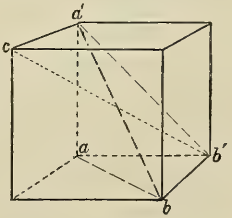


Fig. 233.

In all drawings of this kind the shadows are drawn at an angle of 45° with each co-ordinate plane, and the length of the shadow is made equal to the projection of the object from the

plane on which the shadow is cast. Fig. 234 is the plan of a rectangular prism with its shadow cast on the H.P. The length of the line *ab* determines the height of the prism; so from a single projection of this solid all its dimensions can be ascertained.

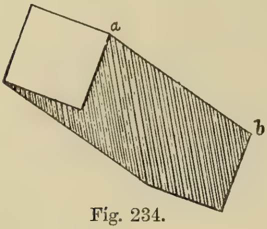


Fig. 234.

The diagonal *cb′* (Fig. 233) is sometimes taken instead of *a′b*; the light is generally assumed to come from the left (exercises excepted). The direction *a′b* is generally used for details, while the direction *cb′* is better suited for drawings illustrating large surfaces, especially elevations.

Note.—The angle *a′ba* (Fig. 233) will be denoted by the sign θ.

It should be understood by the student that making the length of the shadow equal to the projection of the object from the plane on which it is cast is purely conventional. If we refer to Fig. 233 and assume *a′b* to be the direction of light, and *aa′* a line casting a shadow, its shadow ought really to be equal in length to *ab*. Now as *ab* is a diagonal and *aa′* a side of equal squares, *ab* must be longer than *aa′*. In the same way, if *cb′* is taken as the direction of light, the shadow of the line *ca′* ought really to be *a′b′*, which we know must be longer than *ca′*. But the advantage of having a fixed and equal angle for the direction of light on each co-ordinate plane, and to be able to determine the length of the shadow by always making it equal to the projection of the object, so greatly facilitates the work necessary in the projection of shadows, that a convention of this character is admissible.

Frequently, in architectural drawings, the width of the projection casting the shadow is set off on a line at right angles to the projection, instead of along a line at an angle of 45°, as just described. *E.g.* let the side of the cube *ca′a* (Fig. 233) represent a projection from the surface *a′ab*, upon which we wish

to cast its shadow. Set off *ab′* equal to *ca′*, which will be the width of the shadow by this method.

When surfaces perpendicular to each other only are considered, this method answers very well; but it does not give us readily the projection of a line by its shadow, as shown in Fig. 236. Neither is it so well adapted for the projection of curves, etc.

The first method described, and illustrated in the following problems, has many advantages over the latter, which will be more apparent by studying the following illustrations.

Sometimes we have different views of an object on the same sheet of paper, *e.g.* front elevation, side elevation, etc.; but each view, as regards the direction of light, would be treated in the same manner, otherwise, one of the views might be on the side of the object where it would receive no light: the direction of light, in a case of this kind, would be fixed with reference to the plane of the paper, and not with respect to the different views of the object.

In Fig. 235, A is the plan, and A′ the elevation of a cube with one of its faces against the V.P.; B*g* is the shadow on the H.P. of part of the edge *bc*; and *b′*B′ is a shadow on the V.P. of the same edge. This figure shows very plainly the conventional nature of this method of projecting shadows; for we get the shadow of the point B on each coordinate plane, which could not happen if it were a true projection.

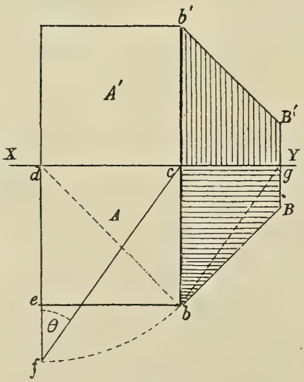


Fig. 235.

Problem 192.

To determine the angle θ.

With *d* as centre (Fig. 235), and radius *db*, draw an arc till it meets *de* produced in *f*. Join *fc*. Then *dfc* is the angle θ.

If we were to set off this angle at *b*, from the line *bc*, it would meet XY in *g*, which would determine the width of the shadow, as shown.

Note.—For practical work in the projection of shadows it is advisable to have a set-square made to this angle (θ), for with the aid of this angle and 45° we are enabled to project all shadows of this nature.

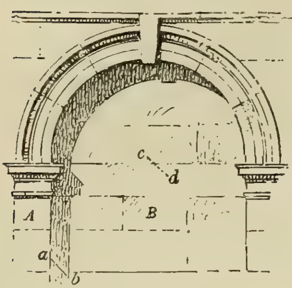


Fig. 236.

Fig. 236 is an arched recess with shadow. Any line in the shadow drawn parallel to *ab* (45°) would show the projection of the face A from the face B. *c* is the centre of the arch; *cd* is drawn at an angle of 45°, and equal in length to the projection of the arch; *d* is the centre for drawing the shadow of the arch.

Problem 193.

The projections of a square being given, to cast its shadow. Fig. 237.

Let *abdc* be the plan, and *a′b′* the elevation of the given square.

The object of this problem is to show the different methods of using the angle θ, *e.g.* we could either set it off at *b* to obtain the point *e*, and draw a projector till it meets a line from *b′* at an angle of 45° in *e′*, or we could set it off at *b′* and obtain the point *f*. Having obtained either of these points, draw a horizontal line from it. Draw a line at an angle of 45° from the point *a′* till it meets

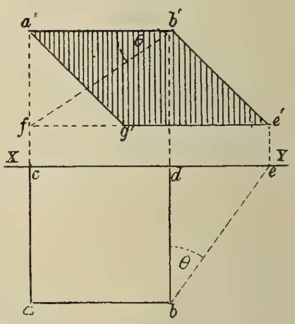


Fig. 237.

this line in the point g'. Then $a'b'e'g'$ is the shadow of the square cast on the V.P.

PROBLEM 194.

The front and side elevations of a console being given, to determine its shadow. Fig. 238.

Let A be the front and B the side elevation of the *console.* Draw any number of horizontal lines to the side elevation. We will take one of these lines (ab) as an illustration. Produce

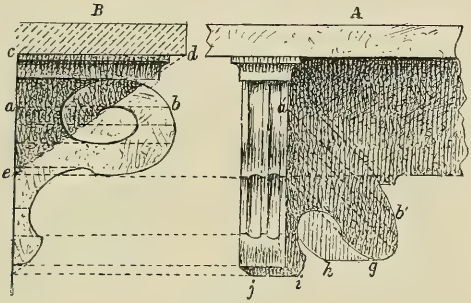


Fig. 238.

ab till it meets the front elevation in a'. Draw $a'b'$ at an angle of 45°, and equal in length to ab. This will determine point b'. Find all the other points in the same way, and draw a fair curve through them.

The shadows of all plane curves could be found in the same way.

After having found the shadow of the side, it is necessary to determine the shadow of the thickness of the console.

Take any points in the lower part of the shadow, *e.g.* the points g and i. From these points draw horizontal lines equal in length to the thickness of the console, as gh and ij, and join them with a fair curve.

This part of the shadow is put in with a lighter shade in the illustration, so as not to interfere too much with the shape

of the shadow of the side, but in reality all would be of the same depth of tone.

The angle *cde* is θ.

Problem 195.

The plan and elevation of a cylinder being given, to determine its shadow. Fig. 239.

Let A be the plan, and A′ the elevation of the cylinder.

Draw the horizontal and vertical diameters of the cylinder,

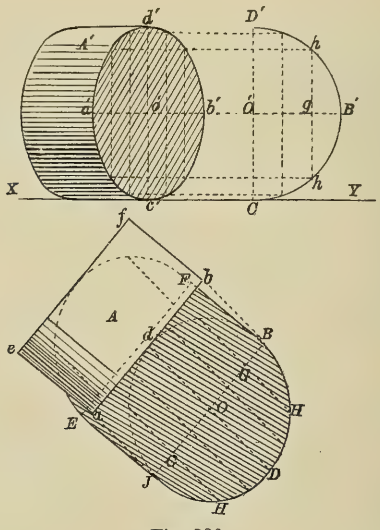


Fig. 239.

as $a'b'$ and $c'd'$, intersecting each other in o'. Produce $a'b'$, and at any convenient point O′ on this line as centre, and with a radius equal to $o'd'$, draw a semicircle D′B′C. Divide O′B′ into any number of equal parts (three), and draw lines through them parallel to the diameter CD′.

From the points a and b on plan draw the lines bB and aJ at an angle of 45° with XY, and equal in length to O′D′. Join JB. This represents the position of the diameter ab projected

into the shadow. As we have divided the radius into three equal parts, we must divide this line (the diameter) into six. Draw lines through each division at an angle of 45° with XY. Make these lines equal in length to the corresponding lines in the semicircle D′B′C, *e.g.* make OD equal to O′D′, GH equal to *gh*, etc. Draw a fair curve through these points.

Next, find the shadow of the length of the cylinder, as follows. Find the position in shadow of the diameter *ef*, as EF. Then find the points in the curve in the same way as described from JB, and join them as shown.

Tangents drawn to these curves, as EJ and FB, will complete the shadow.

If the curves of the elevation are accurately drawn, it is not necessary to draw the semicircle D′B′C. We could divide the diameter *a′b′* into the required number of equal parts, and draw parallel lines as shown, and take the lengths of the lines in the shadow from these lines. Horizontal lines are drawn in the figure, to show that these lines are the same length. This also illustrates another method of projecting circles.

If we wished to project the shadow of a disc instead of a cylinder, we should repeat the lengths of the parallel lines in the shadow, on the opposite side of JB, and join the points, as shown by dotted lines.

Problem 196.

To project the shadow of an octagonal cap on a square prism. Fig. 240.

Draw the lines *ab* and *cd* with the θ set-square (Prob. 192). Draw the projectors *aa′*, *bb′*, etc., till they meet the lines *a′b′* and *c′d′* drawn with the 45° set-square. These give the points necessary for projecting the shadow.

The following shadows are all projected

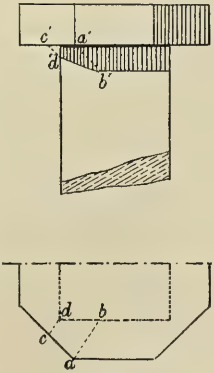


Fig. 240.

in the same way, *i.e.* the lines in the plans are drawn with the θ set-square, and those in the elevation by the 45° set-square. The reason for this will be understood by studying Fig. 235.

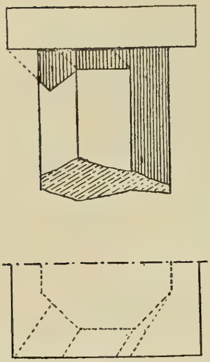


Fig. 241.

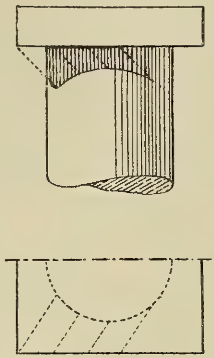


Fig. 242.

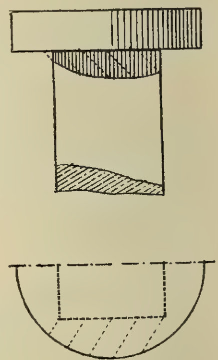


Fig. 243.

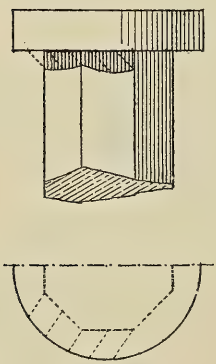


Fig. 244.

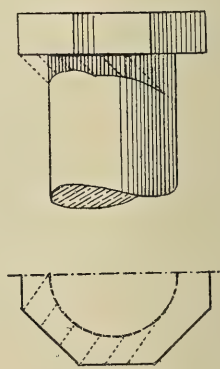


Fig. 245.

Fig. 241 is the shadow of a square cap on an octagonal prism.
„ 242 „ „ „ „ a cylinder.
„ 243 „ „ circular „ a square prism.
„ 244 „ „ „ „ an octagonal prism.
„ 245 „ „ an octagonal „ a cylinder.

Problem 197.

To project the shadow of a circular flange upon a cylinder. Fig. 246.

In this instance the side elevation A answers the same purpose as the plans in the preceding examples.

Any number of points may be taken in the circle forming the flange. We will take one point as an illustration, *a* for instance. Draw the line *ab* with the θ set-square. Draw the horizontal line *aa′*. From *a′* draw the line *a′b′* with the 45° set-square till it meets a horizontal line from *b* in *b′*. Find all the other points in the same manner, and join them with a fair curve.

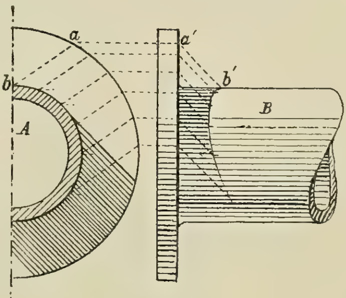


Fig. 246.

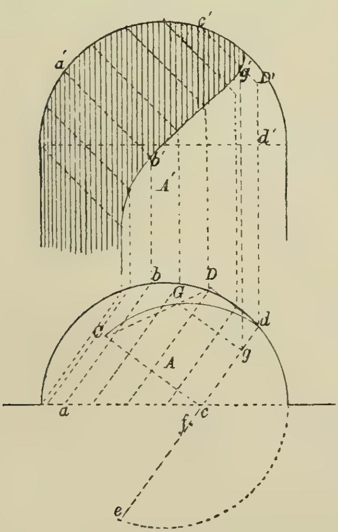


Fig. 247.

Problem 198.

To project the shadow cast on the interior of a semicircular recess. Fig. 247.

Let A be the plan, and A′ the elevation.

Take any number of points in the arch over the recess, as *a′*, *c′*, etc., and find the corresponding points in the plan, as *a*, *c*, etc. Draw the line *ab* with the θ set-square. Draw the projector *bb′*; and from *a′*, with the 45° set-square, draw a line till it meets *bb′* in *b′*. All the

points in the shadow below the soffit, or curved top, are determined in the same manner.

Draw the line *cd* with the θ set-square, and draw the projector *dd′*.

We must now "construct" on to the H.P. the curve of the soffit of the recess in a plane on the line *cd*.

We know that all sections through a sphere are circles, and as the soffit of the recess is a quarter of a sphere, the curve we wish to find must be the arc of a circle. Continue the plan of the recess till it meets *cd* produced in *e*. Then *cd* is the plan of the section through a sphere containing the arc required. Bisect *ed* in *f*. With *f* as centre, and radius *fd*, draw an arc till it meets a perpendicular on *c* in C. This is the arc "constructed" on a H. plane.

From *c′* draw a line at an angle of 45° till it meets *dd′* produced in D′. Draw the line *d*D perpendicular to *ed*, and equal in length to *d′*D′. Join CD, cutting the arc C*d* in G. Draw the line G*g* perpendicular to *ed*. Draw the projector *gg′*, meeting the line *c′*D′ in *g′*. Find the other points in the soffit in the same way, and join them to complete the shadow.

PART II.—SHADOWS OF OBJECTS CAST BY PARALLEL RAYS OF LIGHT AT GIVEN ANGLES

Problem 199.

The plan and elevation of a cube, with the direction of light, being given, to determine the shadow on the HP. Fig. 248.

Let D and D′ be the projections of the cube, and R and R′ the projections of a given ray of light.

Draw lines from the points *a′*, *c′*, and *b′*, parallel to the ray R′, till they meet XY in the points A′, C′, and B′. Draw lines from the points *a*, *b*, and *c*, parallel to R, till they meet projectors

drawn from the points A′, C′, and B′, in the points A, C, and B. Join AB and BC, to complete the shadow.

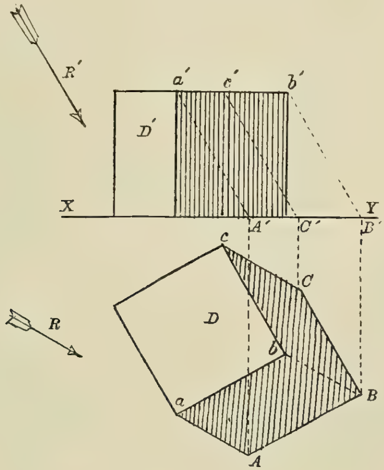


Fig. 248.

The two sides of the cube turned away from the light, as *ab* and *bc*, are said to be in shade.

ORTHOGONAL PROJECTION.

In the preceding chapter (*Horizontal Projection*) it has been shown how all the dimensions of an object can be determined from one of its projections, viz. a plan, providing we have the heights given of its several parts.

In a similar manner it is possible to determine the elevation or height of an object from its plan and shadow, if the inclination of the given ray of light to the H.P. is known. The shadow, in fact, is another projection of the object on the same plane, and when this is determined by parallel rays of light, it is called *Orthogonal Projection.*

In Fig. 249 we have the plan and shadow of a pyramid, in which A is the shadow or second projection of the vertex *a* on the H.P.

Let the inclination of the ray R with the H.P. be 60°.

Join A*a*, and assume it to be a ground line. Set off at A, with A*a*, the angle 60° till it meets a perpendicular on *a* in *a*′. Then *a*′ is the elevation of the vertex *a*. With *a* as centre, and radius *ab*, draw an arc till it meets A*a* in B. Join B*a*′, which is the true length and inclination of the edge *ba* of the pyramid.

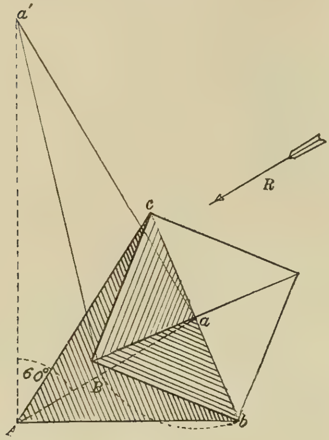


Fig. 249.

The Line of Separation.—When rays of light coming from the same source or direction are intercepted by a solid, the part facing the light is illuminated, while the part turned from the light is in shade. The division between these two parts is called the *line of separation*, *e.g.* the line *cab* (Fig. 249) is the line of separation between the light and shade on the pyramid.

In cylinders and cones this line is similar to the line of intersection between a tangent plane and the solid. The T plane would contain a line parallel to the given ray.

Problem 200.

To determine the shadow and line of separation of a given right vertical cone, the direction of light being given. Fig. 250.

Let B and B′ be the plan and elevation of the cone, and R and R′ the projections of the given ray.

Draw a line from the vertex *a*′ parallel to R′ till it meets XY in A′. Draw a line from *a* parallel to R till it meets a projector from A′ in A.

Draw A*b* and A*c* tangential to the base of the cone, and join *ba* and *ca*. Then *b*A*c* is the shadow of the cone, and *bac* the line of separation.

Draw the projector bb', and join $b'a'$. This is the elevation of the line of separation.

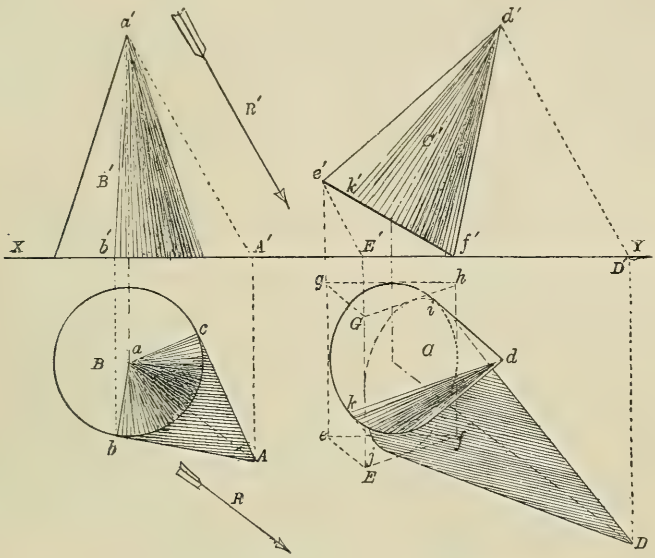


Fig. 250. Fig. 251.

Problem 201.

To determine the shadow and line of separation of a given right cone with its base inclined to the H.P., *the direction of light being given.* Fig. 251.

Let C and C′ be the plan and elevation of the cone, and R and R′ the projections of the given ray.

Enclose the plan of the base of the cone in a rectangle parallel to XY, as *gefh*. Draw *e*′E′ parallel to R′, and from E′ draw a projector till it meets a line from *e* drawn parallel to R in E.

Draw a line from *g* parallel to R till it meets E′E in G. Join G*h* and E*f*. Then GE*fh* is the shadow of the rectangle *gefh*. Inscribe an ellipse in GE*fh* (Prob. 185, Plane Geometry). This is the shadow of the base of the cone.

Draw a line from the vertex *d*′ parallel to R′ till it meets XY

in D′. Draw a projector from D′ till it meets a line drawn from *d* parallel to R in D. Draw tangents from D to the ellipse forming the shadow of the base (Prob. 183, Plane Geometry), as D*i*, D*j*. These give the shadow of the cone on the H.P.

From *j* draw a tangent to the base of the cone, as *jk*. Join *kd*. Draw a projector *kk′*, and join *kd′*. Then *kd* is the plan, and *k′d′* the elevation of the line of separation.

Problem 202.

To determine the shadow and line of separation of a given right vertical cylinder, the direction of light being given. Fig. 252.

Let A and A′ be the plan and elevation of the cylinder, and R and R′ the projections of the given ray.

From the top point of the axis (*c′*) draw a line parallel to R′ till it meets XY in C′. Draw a projector from C′ till it meets a line drawn from *c* parallel to R in C.

Draw a diameter to the plan A perpendicular to *c*C, as *de*. Draw a line through C parallel to *de*, as *fg*. With C as centre, and radius equal to *cd*, draw a semicircle. Join *df* and *eg*, which complete the shadow of the cylinder.

Draw a projector from *d*, which gives *d′h*, the line of separation.

Problem 203.

To determine the shadow and line of separation of a given cylinder inclined to both planes of projection, the direction of light being given. Fig. 253.

Let B and B′ be the plan and elevation of the cylinder, and R and R′ the projections of the given ray.

Enclose the plan of the cylinder in a rectangular prism by drawing the lines *op* and *mn* perpendicular to its sides produced and tangential to its base. Proceed in the same manner with

the plan of the top of the cylinder by drawing the lines *ij* and *kl*. Find the elevation of this rectangular prism.

First find the outline of the shadow to the plan of the base of this prism, by drawing lines from *o′* and *p′* parallel to R′ till they meet XY in O′ and P′. Draw lines from *o* and *p* parallel

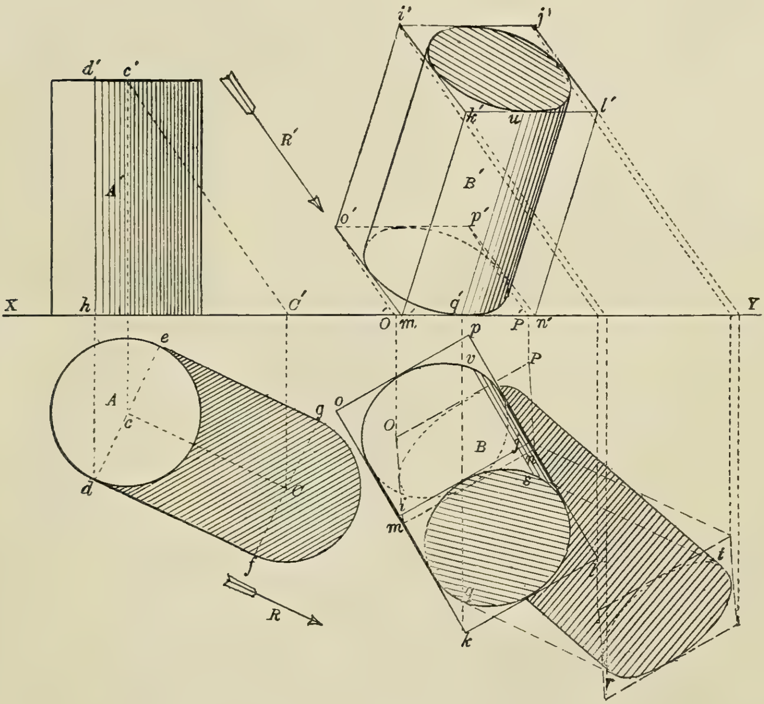


Fig. 252. Fig. 253.

to R till they meet projectors from O′ and P′ in O and P. Join OP, O*m*, and P*n*, and inside this parallelogram inscribe an ellipse (Prob. 185, Plane Geometry). Proceed in the same manner with the plan of the top of the prism, and draw tangents to the two ellipses thus found to complete the shadow.

Draw tangents to the ellipses forming the plan of the top of the cylinder and its shadow, as *qr* and *st*. Draw the projector

qq', and draw $q'u$ parallel to the line $m'k'$. Then $q'u$ is the line of separation in the elevation. From s draw sv parallel to the side of the cylinder. This is the plan of the other line of separation.

Problem 204.

To determine the shadow of a given hexagonal prism, inclined to both planes of projection, the direction of light being given. Fig. 254.

Let A and A′ be the plan and elevation of the prism, and R and R′ the projections of the given ray.

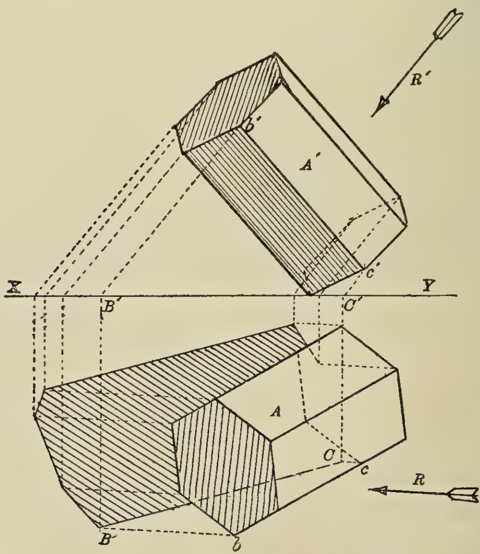


Fig. 254.

Draw lines from b' and c' parallel to R′ till they meet XY in B′ and C′. Draw lines from b and c parallel to R till they meet projectors from B′ and C′ in B and C. Find the shadows of the other edges in the same way, and join them as shown.

Problem 205.

To determine the shadow on both planes of projection, and line of separation, of a given vertical octagonal cone, the direction of light being given. Fig. 255.

Let A and A′ be the plan and elevation of the solid, and R and R′ the projections of the given ray.

First determine the H. and V. traces of a ray passing through

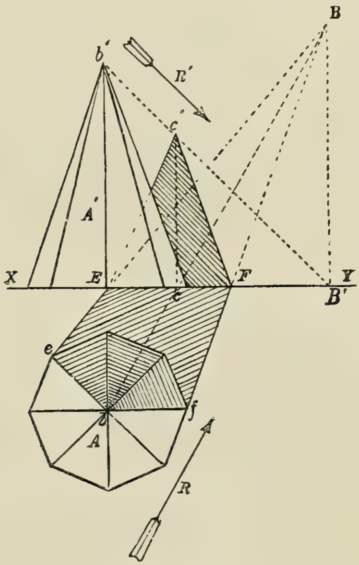


Fig. 255.

the vertex, as follows. Draw a line from b' parallel to R′ till it meets XY in B′. Draw a line from b parallel to R till it meets a projector on B′ in B, and cutting XY in c. Draw a projector from c till it meets b'B′ in c'. Then B is the H.T., and c' the V.T. required.

Draw lines from B till they meet the plan A in e and f, cutting XY in E and F. Join c' to E and F. The shadow is then completed. *ebf* is the line of separation.

Problem 206.

To determine the shadow on both planes of projection of a given rectangular prism inclined to both co-ordinate planes, the direction of light being given. Fig. 256.

Let A and A′ be the plan and elevation of the prism, and R and R′ the projection of the given ray.

Draw a line from b' parallel to R′ till it meets XY in B′.

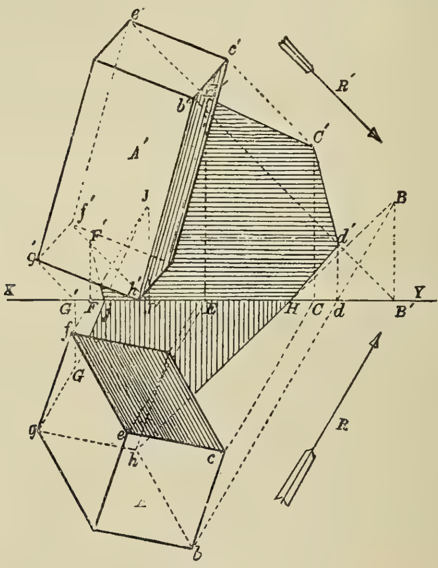


Fig. 256.

Draw a line from b parallel to R, cutting XY in d, till it meets a projector on B′ in B. Draw a projector from d till it meets b'B′ in d'. Then B is the H.T., and d' the V.T. of a line from b parallel to the given ray. Join hB by a line cutting XY in H. Then hB is the H.T. of a plane containing the edge bh, and a line parallel to the given ray. Join Hd'. Draw lines from c and e parallel to R till they meet XY in C and E. Draw lines from c' and e' parallel to R′ till they meet projectors from C and E in C′ and E′. Join E′C′ and C′d'.

Draw a line from f' parallel to R′ till it meets XY in J′; also from f parallel to R till it meets a projector from J′ in J, and cutting XY in F. Draw a projector from F till it meets f'J′ in F′. Draw a line from g' parallel to R′ till it meets XY in G′, also from g parallel to R till it meets a projector from G′ in G. Join GJ, cutting XY in j. This is the H.T. of a plane containing the edge fg and a line parallel to the given ray. Join jF′ to complete the shadow.

PROBLEM 207.

To project the shadow of a given disc on both planes of projection; the disc to be inclined to the H.P., *the direction of light being given.* Fig. 257.

Let A and A′ be the plan and elevation of the disc, and R

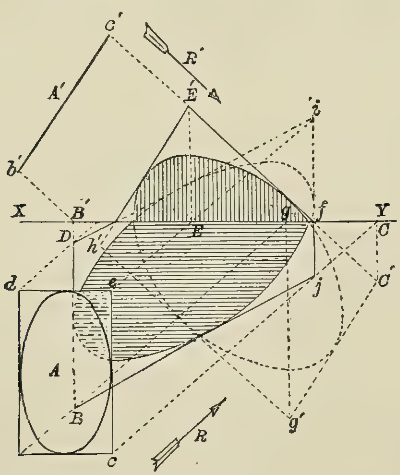


Fig. 257.

and R′ the projections of the given ray. Enclose the plan in a rectangle parallel to the V.P., as *dbce*.

First project an outline of the shadow of the entire rectangle on the H.P., also on the V.P., as follows:—

Draw a line from c parallel to R till it meets XY in C.

Draw a line from c' parallel to R′, cutting XY in f, till it meets a projector from C in C′. Draw a perpendicular to XY from f till it meets cC in j. Draw a line from e parallel to R till it meets jf produced in i'.

Draw a line from b' parallel to R′ till it meets XY in B′. Draw lines from b and d parallel to R till they meet a perpendicular from B′ in B and D. Join Di and Bj. This is the shadow of the rectangle on the H.P.

From the point E, where the line ei cuts XY, draw a perpendicular till it meets c'C′ in E′; and where the line bB produced meets XY in g, draw a perpendicular till it meets b'B′ produced in g'. Join C′g'. Also draw the line E′h' parallel to C′g'. This completes the shadow of the rectangle on the V.P.

In each of these rectangles inscribe ellipses (Prob. 185, Plane Geometry), and complete the shadow of the disc as shown.

Problem 208.

To project the shadow of a given vertical rectangular prism on a given cylinder, with the shadows of both objects on the H.P., *also the line of separation, the direction of light being given.* Fig. 258.

Let A and A′, B and B′, be the plans and elevations of the two solids; and R and R′ the projections of the given ray.

Enclose the elevation of the cylinder in a rectangular prism.

Draw lines from c' and d' parallel to R′ till they meet XY in the points C′ and D′. Draw lines from the points c and d parallel to R till they meet projectors from C′ and D′ in C and D. Join CD. This is the outline of the shadow of one end of the prism enclosing the cylinder on the H.P. Proceed in the same manner with the opposite end, and draw ellipses in them (Prob. 185, Plane Geometry). Draw tangents to these ellipses to complete the shadow of the cylinder on the H.P.

Draw a line from e' parallel to R′ till it meets XY in E′. Draw a line from e parallel to R till it meets a projector from

E′ in E. Find the other points of the vertical prism in the same manner, and join them, as shown, to complete its shadow on the H.P.

In any convenient position draw the end elevation of the cylinder, as F′, and draw the diameter $g'h$ perpendicular to XY. Draw a line from g' parallel to R′ till it meets XY in i', also a

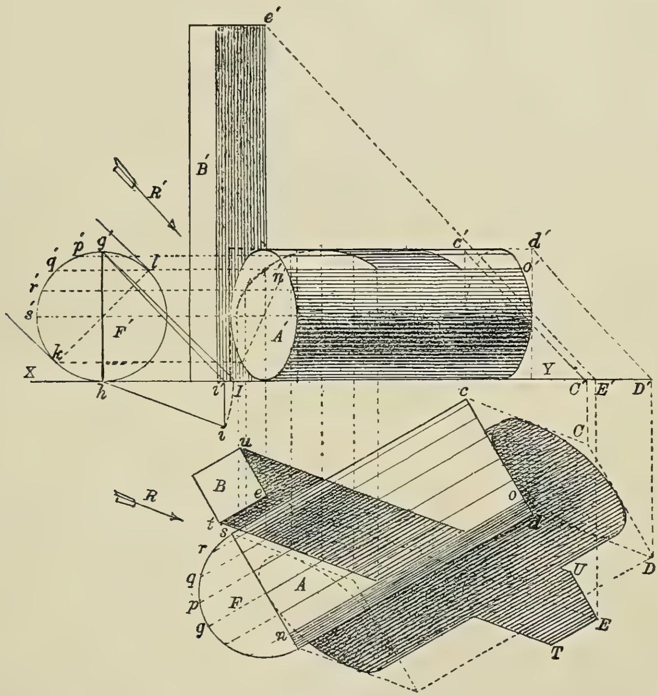


Fig. 258.

line from h parallel to R till it meets a projector from i' in i. With h as centre, and radius hi, draw an arc till it meets XY in I. Join g'I. Then $g'\mathrm{I}h$ is the true inclination of the ray R to the H.P. Draw the diameter kl perpendicular to g'I. These are the points in which tangents parallel to the given ray would touch the cylinder. Draw a line from l parallel to XY till it meets the ends of the cylinder in n' and o'. This is the line of separation in elevation, *i.e.* the line of contact between the cylinder

and a tangent plane containing a line parallel to the given ray. The other line of separation, on the opposite side of the cylinder, is determined by drawing a horizontal line from k.

Draw projectors from n' and o' to obtain n and o in plan, and join them to obtain the H. projection of the line of separation.

Draw the semicircle F at the end of the plan of cylinder, and divide it into any number of equal parts, as g, p, q, etc. Also divide the upper part of the circle F′ into the same number of parts, lettered to correspond, as g', p', q', etc. Draw lines from each of these points, both in plan and elevation, parallel to the axis of the cylinder. Where these parallels in plan meet the lines tT and uU of the shadow of the vertical prism, draw projectors till they meet the corresponding parallels in elevation; and draw fair curves through the points of intersection, to complete the shadow of the vertical prism on the cylinder, in elevation.

Note.—The projectors for one curve only are shown, to avoid confusion.

Problem 209.

To determine the projections of the line of separation on a given sphere, with the shadow of the sphere on the H.P., *the direction of light being given.* Fig. 259.

Let A and A′ be the projections of the given sphere, and R and R′ those of the given ray.

First determine a line from the centre of the sphere, parallel to the V.P., making the same angle with the H.P. as the given ray, as follows. Draw a line from the centre of A′ parallel to R′ till it meets XY in C′. Draw a projector from C′ till it meets a line from the centre of A parallel to R in C. With c as centre, and radius cC, draw an arc till it meets a horizontal line through c in d. Draw the projector dd', and join $d'c'$, which is the elevation of the line required.

Through c' draw $e'f'$ perpendicular to $c'd'$. If the ray of light were parallel to the V.P., then $e'f'$ would be the elevation of the

circle forming the line of separation on the sphere, in a plane perpendicular to the direction of light. To find this line in its proper position, produce it till it meets XY in g'. Draw the projector $g'g$. With c as centre, and radius cg, draw an arc till

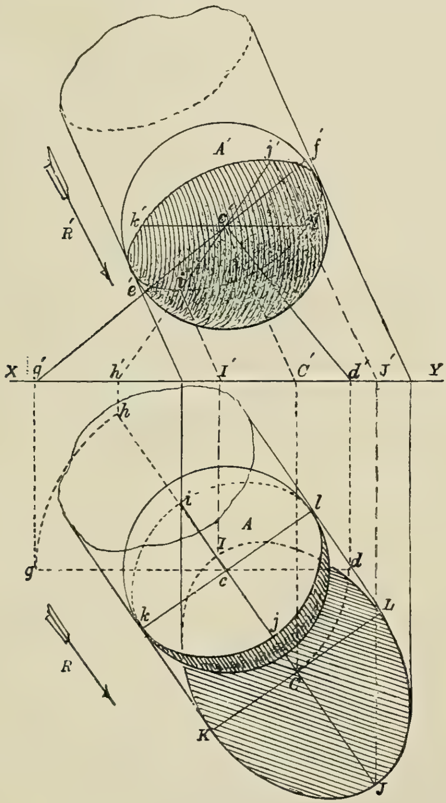


Fig. 259.

it meets Cc produced in h. Draw the projector hh', and draw a line from h' through c' till it meets horizontal lines drawn from e' and f' in the points i' and j'. Draw projectors from i' and j' till they meet Ch in i and j. Through c draw kl perpendicular to ij. Draw projectors from k and l till they meet a horizontal line through c' in k' and l'. Then kl and ij, $k'l'$ and $i'j'$, are the axes of the ellipses forming the plan and elevation of the line of

separation on the sphere. Draw the ellipses (Probs. 181 and 185, Plane Geometry).

To draw the shadow of the sphere on the H.P. Draw a line through C perpendicular to cC till it meets lines drawn from k and l parallel to cC in the points K and L.

Draw lines from i' and j' parallel to R′ till they meet XY in I′ and J′. Draw projectors from I′ and J′ till they meet cC produced in I and J. Then IJ is the major, and KL the minor axes of the ellipse forming the shadow. Draw the ellipse.

Note.—This problem also shows the projections of a given sphere, enveloped by an oblique cylinder, with its axis at a given angle. The line of contact between the sphere and cylinder would be the line of separation between the light and shade on the sphere, and the axis of the cylinder would be parallel to the given ray. The shadow of the sphere on the H.P. would be the base of the oblique cylinder. The projections of the cylinder are indicated by faint lines.

Problem 210.

The plan and elevation of an icosahedron, with the direction of light, being given, to determine the projection of its shadow on the H.P. Fig. 260.

Let A and A′ be the plan and elevation of the solid, and R and R′ the projections of the given ray.

Draw lines from the points e', a', d', b', and c', in the elevation, parallel to the arrow R′, till they meet XY in the points E′, A′, D′, B′, and C′.

Draw lines from the corresponding points in the plan, parallel to the arrow R, till they meet perpendiculars drawn from the corresponding letters on XY. These give the points A, B, C, D, and E. Join them as shown. Also join A to g, and E to f, to complete the shadow.

PROBLEM 211.

The plan and elevation of the group of objects (Fig. 261), *with the direction of light, being given, to determine the projection of their shadows.*

The points of the shadows of the dodecahedron, and the

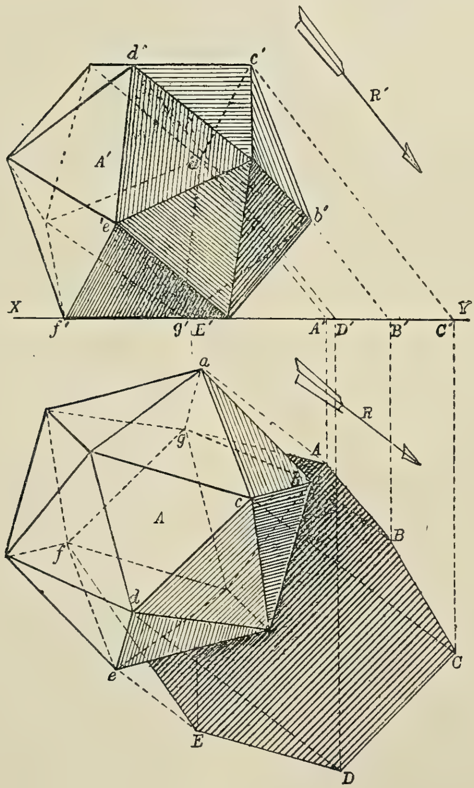


Fig. 260.

rectangular prism on the H.P., are determined as described in the preceding problem.

Some of the points of the shadow of the dodecahedron are

intercepted by the rectangular prism, *e.g.* a line drawn from b'

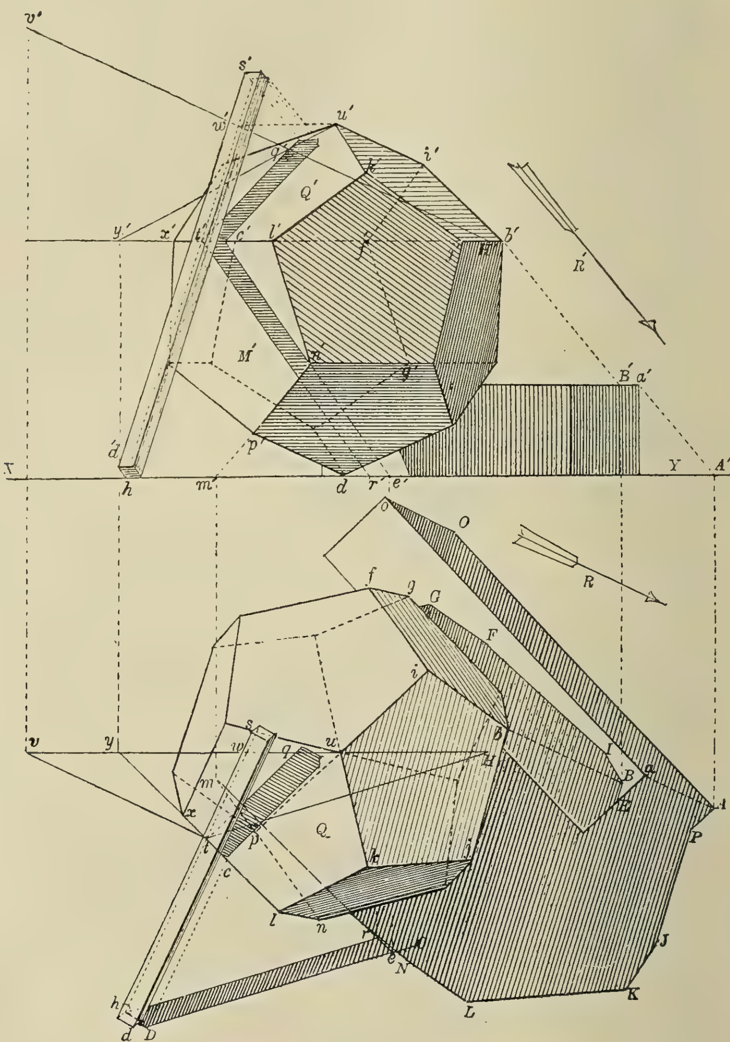


Fig. 261.

parallel to R′ will meet the top surface of the prism in B′. Draw a line from b parallel to R till it meets a projector from

B′ in B. The points G, F, and I are found in the same manner.

Draw the line from BE parallel to the line PJ. The short line from G is found in a similar manner.

The shadow of the rod on the dodecahedron is found in the following manner:—

First determine the shadow of the edge TS of the rod on the face Q of the solid. The shadow of this line is the line of intersection of two planes : one plane will contain a line parallel to the given ray and the line TS, and the other plane is the face Q of the solid. Determine the traces of these planes in the following manner.

Note.—As the traces of these planes, if taken on the H.P. and V.P., would come beyond the limits of the drawing, we can assume supplementary co-ordinate planes; *e.g.* let a line drawn through u parallel to XY, as vu, be the H.T. of a V.P., and a line drawn at the level of $x'l'$ be the V.T. of a H.P.

Draw a line from t parallel to the ray R till it meets vu in v. The elevation of v, as v', is determined by a line from t' parallel to R′. From the point w, where the line ts meets the line vu, draw a projector till it meets the line $t's'$ in w'.

Draw a line from v' through w' till it meets $x'l'$ produced in H′. Find the plan of H′, as H, and draw a line from H through t. Then Ht is the H.T., and H′v' the V.T. of a plane containing a line parallel to the given ray and the edge TS of the rod.

Produce the line lx till it meets vu in y. Draw a projector from y till it meets $l'x'$ produced in y'. Join $y'u'$. This is the V.T. of the face Q′ of the solid.

From the point q', where the two V.T.s intersect, draw a line to t'. Find the plan of q', as q, and join qt. Then qt is the plan, and $q't'$ the elevation of the line of intersection between the two planes, and is therefore the shadow of the line TS on the face Q of the solid.

To find the shadow of the rod on face M′ of the solid. Produce $n'p'$ till it meets XY in m'. Find the plan of m', as m,

by producing *np* till it meets a projector from *m′*. Draw a line from *m* parallel to *xl*, as *me*. This is the H.T. of the face M. Find the shadow of the point *d*, as D. From D draw D*c* parallel to the rod. Find the elevation of *c*, as *c′*. Find the shadow of point *c*, as C, cutting the H.T. of the face M in *e*. Draw a line from *h* parallel to D*e*, as *hr*. These two lines form the shadow of the rod on the H.P.

Draw projectors from *r* and *e* till they meet XY in *r′e′*. Join *c′e′* and *t′r′*. These give the shadow on face M′. Draw a line from *c′* parallel to *t′q′*, and draw lines from the top of the rod parallel to the ray R′ till they meet these lines, to complete the shadow on face Q′. Complete the plan of the shadow on face Q in the same manner.

Note.—The shadows of lines cast upon solids could be illustrated by all the problems in Chap. XXX. If the cutting plane in each instance contained the line that casts the shadow and a line parallel to the given ray, the edge of the section facing the direction of light would represent the shadow of the line upon the solid.

To determine the traces of a plane containing two given lines not parallel to each other, see Prob. 115.

If the shadow of a cone or cylinder were cast upon another solid, the shadow would be the intersection of the solid by tangent planes to the cone or cylinder containing lines parallel to the given ray. To determine the traces of a tangent plane parallel to a given line, see Probs. 143 and 147.

PART III.—SHADOWS OF OBJECTS CAST FROM A LUMINOUS POINT

The principle of construction for finding the various points of the shadows in this Part is the same as that illustrated in Part II., only the lines that determine these points are drawn

radiating from the projections of the *luminous point*, instead of parallel to the projections of the given *ray*.

A few problems will suffice to illustrate this principle.

Problem 212.

To determine the shadow of a given square on the H.P., *the projections of the luminous point being given.* Fig. 262.

Let A and A′ be the projections of the given square, and L and L′ the projections of the luminous point.

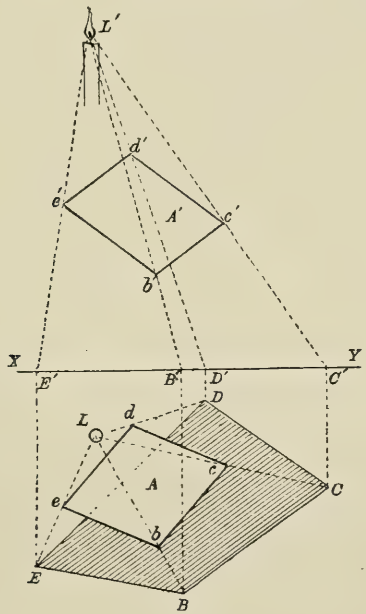


Fig. 262.

Draw lines from L′ through the points e', b', c', d' till they meet XY in the points E′, B′, D′, C′.

Draw lines from L through the points e, b, c, d till they meet projectors from the points E,′ B,′ D,′ C′ in the points E, B, D, C. Join these points, as shown.

Problem 213.

To determine the shadow of the same square on the V.P., *the projections of the luminous point being given.* Fig. 263.

Draw lines from L through the points *e*, *b*, *c*, *d* till they meet XY in the points E, B, D, C.

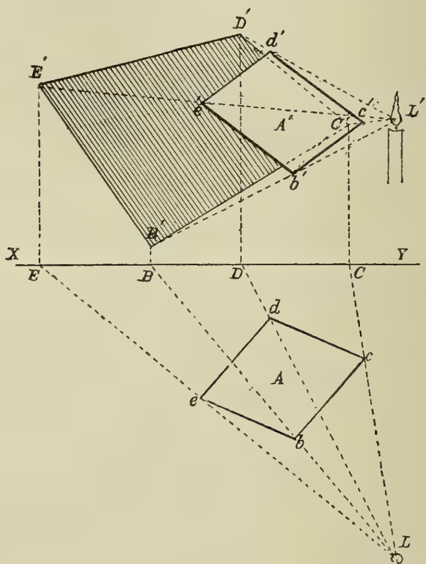


Fig. 263.

Draw lines from L′ through the points *e*′, *b*′, *c*′, *d*′ till they meet projectors from the points E, B, D, C in the points E′, B′, D′, C′. Join these points, as shown.

Problem 214.

To determine the shadow of the same square on both planes of projection, the projections of the luminous point being given. Fig. 264.

Find the points B and D′, as explained in the two preceding problems.

Find the H.T. of the ray through c in the following manner: Draw a line from L′ through c' till it meets XY in g'. Draw

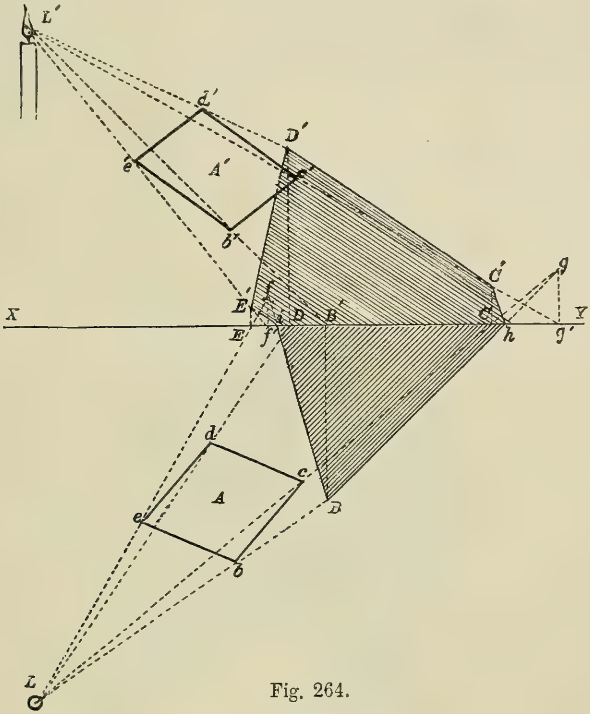


Fig. 264.

a line from L through c till it meets a projector on g' in g, and cutting XY in C. Then g is the H.T. of a line through c. Draw a projector from C till it meets $c'g'$ in C′. Join Bg, cutting XY in h. Join hC′ and D′C′. Find the H.T. of a line through e, as f, in the same manner, and determine the points i and E′. Join the points, as shown, to complete the shadow.

Problem 215.

To determine the shadow of a given octahedron on both planes of projection, the projections of the luminous point being given. Fig. 265.

Let A and A′ be the projections of the given solid, and L and L′ the projections of the luminous point.

Find the points D and G′, as described in the Problems 212 and 213.

Find the V.T. of a line through *e* in the following manner: Draw a line from L through *e* till it meets XY in *i*. Draw

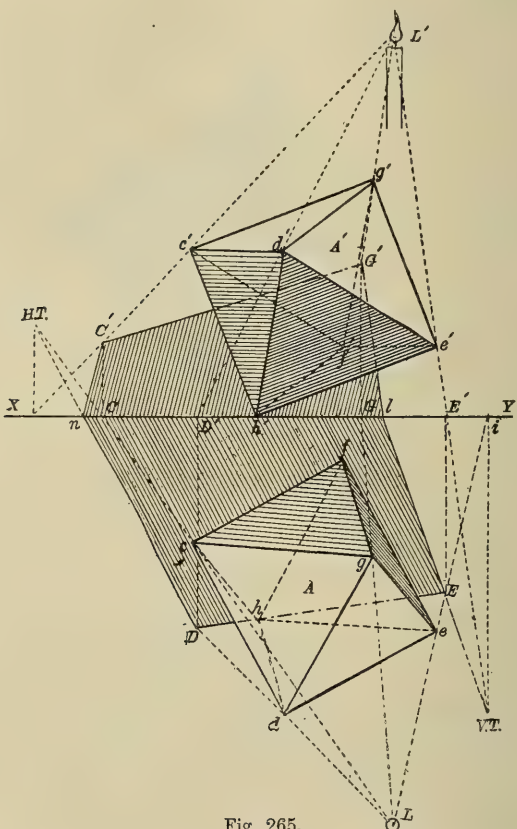


Fig. 265.

a line from L′ through *e*′, cutting XY in E′, till it meets a projector from *i*. This is the V.T. required. Draw a projector from E′ till it meets *ei* in E. Draw a line from the V.T. through E till it meets XY in *l*. Join *l*G′. Join D*h* and E*h*.

Find the H.T. of a line through *c*, as described in the preceding problem, and determine points *n* and C′. Join the points, as shown, to complete the shadow.

Problem 216.

To determine the line of separation and the shadow of a given sphere on the H.P., *the projections of the luminous point being given.* Fig. 266.

Let A and A′ be the projections of the given sphere, and L and L′ those of the given luminous point.

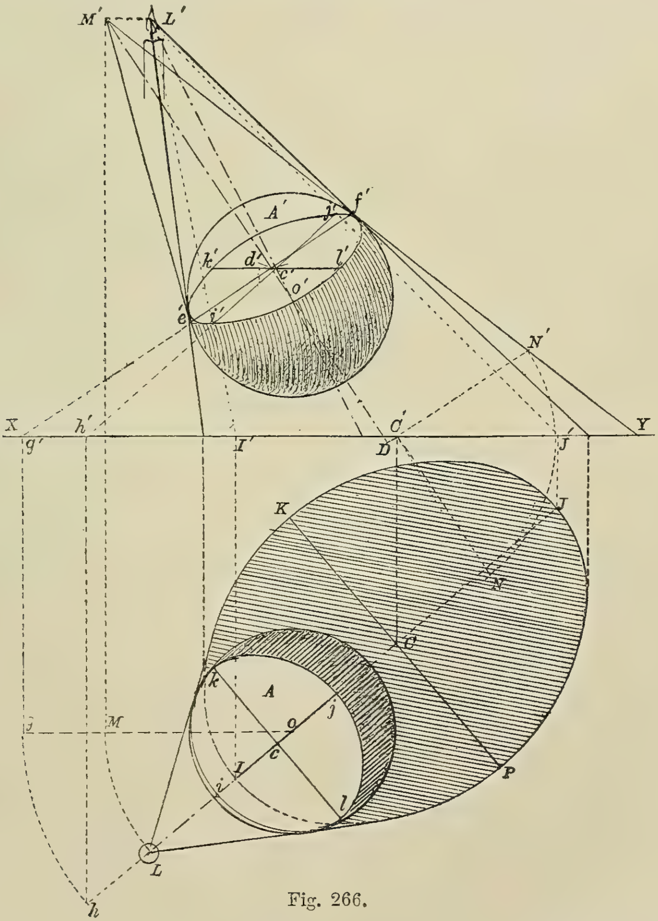


Fig. 266.

First determine a line from the centre of the sphere to the

luminous point, "rotated" till it is parallel to the V.P., as follows:—

Draw a line from o, the centre of the plan of the sphere, parallel to XY, as og. With o as centre, and radius oL, draw an arc till it meets og in M. Draw a projector from M till it meets a horizontal line from L′ in M′. Join M′o', which is the line required.

From M′ draw lines tangential to A′, and touching it in the points e' and f'. Join $e'f'$ cutting M′o' in d': $e'f'$ should be perpendicular to M′o'. M′$e'f'$ is the side elevation of a cone. Through d' draw a horizontal line cutting L′o' in c'.

We will next "rotate" this cone till its vertex coincides with the luminous point. Produce $e'f'$ till it meets XY in g'. Draw a projector from g' till it meets og in g. With o as centre, and radius og, draw an arc till it meets oL produced in h. Draw the projector hh'. Draw a line from h', through c', till it meets horizontal lines drawn from e' and f' in i' and j'.

Join Lo. Draw a projector from c' till it meets Lo in c. Draw projectors from i' and j' till they meet Lo produced in i and j. Draw a line through c perpendicular to Lo. On this line set off from c, cl and ck, each equal to $d'f'$. Draw projectors from k and l till they meet the horizontal line through c' in k' and l'. Then ij, kl, and $i'j'$, $k'l'$, are the axes of the ellipses forming the projections of the circles, which are the lines of separation of the light and shade on the surface of the sphere. Draw these ellipses (Probs. 181 and 185, Plane Geometry).

To determine the shadow of the sphere on the H.P. Draw lines from L′ through the points i' and j' till they meet XY in the points I′ and J′. Draw projectors from I′ and J′ till they meet Lo produced in I and J. Bisect IJ in C, and draw the projector CC′. Produce the axis M′o', and draw a line through C′ perpendicular to it, and meeting it in D, and M′f' produced in N′. With D as centre, and radius DN′, draw an arc till it meets a perpendicular to DN′ from C′ in the point N.

Draw a line through C at right angles to I and J, and set off

on it from C, K, and P, each equal to C′N. Then IJ is the major, and KP the minor axis of an ellipse forming the shadow of the sphere on the H.P. Draw the ellipse (Prob. 181, Plane Geometry).

This problem also illustrates an oblique cone, with the inclination of its axis at a given angle, enveloping a given sphere. The line of separation is the circle of contact; and a line passing from the luminous point through the centre of the sphere, the given inclination of its axis. The shadow of the sphere on the H.P. is the base of the cone.

CHAPTER XXXIV

INSCRIBED SOLIDS

A SOLID is said to be *inscribed* when all its angular points are in contact with the solid circumscribing it.

The five regular solids (Chap. XIX.) can be inscribed in a sphere, as there is a point in them from which all the angular points are equidistant.

Right cones and pyramids can be inscribed in a sphere, and an octahedron in a cube, etc.

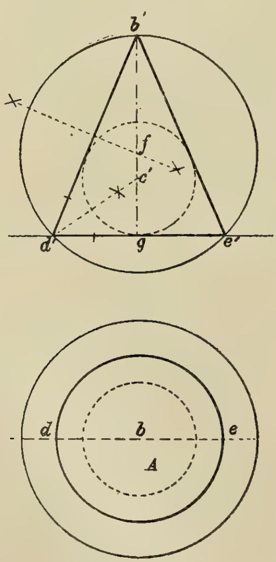


Fig. 267.

All these solids can likewise circumscribe a sphere, when their surfaces are tangential to the surface of the inscribed sphere.

PROBLEM 217.

1. *To inscribe a given cone in a sphere.*
2. *To inscribe a sphere in a given cone.*

Fig. 267.

Let A and A′ be the projections of the given cone. Draw the axis *b′g*, and the plan of the vertex, *b*.

1. Bisect the line *d′b′* by a perpendicular cutting the axis in *f*. With *f* as centre, and radius *fb′*, draw a circle. With *b* as centre,

draw another circle of the same radius. These are the projections of the circumscribed sphere required.

2. Bisect the angle $b'd'e'$ by a line cutting the axis in c'. With c' as centre, and radius $c'g$, draw a circle. With b as centre, draw a similar circle. These are the projections of the inscribed sphere.

Problem 218.

1. *To inscribe a given hexagonal pyramid in a sphere.*
2. *To inscribe a sphere in a given hexagonal pyramid.* Fig. 268.

Let A and A′ be the projections of the pyramid.

1. Draw the axis $b'g$, and the plan of the vertex, b.

With b as centre, and one of the angular points (e) as radius, draw an arc till it meets a horizontal line through b in d. Draw the projector dd', and join $d'b'$.

Bisect $d'b'$ by a perpendicular cutting the axis in f. With f as centre, and radius fb', draw a circle. With b as centre, and with the same radius, draw a similar circle. These circles are the projections of the circumscribing sphere.

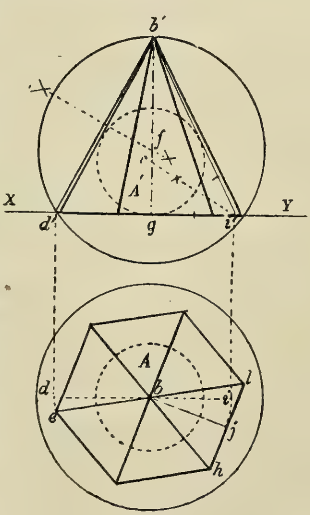


Fig. 268.

2. Bisect one of the sides, hl for instance, in the point j. With b as centre, and radius bj, draw an arc till it meets db produced in i. Draw the projector ii', and join $i'b'$.

Bisect the angle $b'i'g$ by a line cutting the axis in c. With c as centre, and radius cg, draw a circle; and with b as centre, draw a similar circle. These are the projections of the inscribed sphere.

Note.—The same construction would apply to all the regular pyramids.

Problem 219.

1. *To inscribe a given tetrahedron in a sphere.*
2. *To inscribe a sphere in a given tetrahedron.* Fig. 269.

The construction of this problem is the same as the one pre-

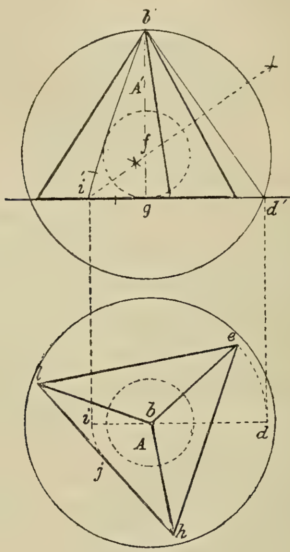


Fig. 269.

ceding, and as the different parts bear similar letters, the same text will apply to both.

Note.—The inscribed and circumscribing spheres in this instance are concentric.

Problem 220.

To project an octahedron in a similar position to a given octahedron, but inscribed in a sphere one inch in diameter. Fig. 270.

Let A and A′ be the projections of the given octahedron. First inscribe this solid in a sphere.

We know that each of its angular points are equidistant

from a point which must be the centre of the circumscribing sphere. To determine this point join any two opposite points, b'I$'$ for instance, and bisect the line in c'. Serve the plan in a similar manner. Determine the true length of $b'c'$ in the follow-

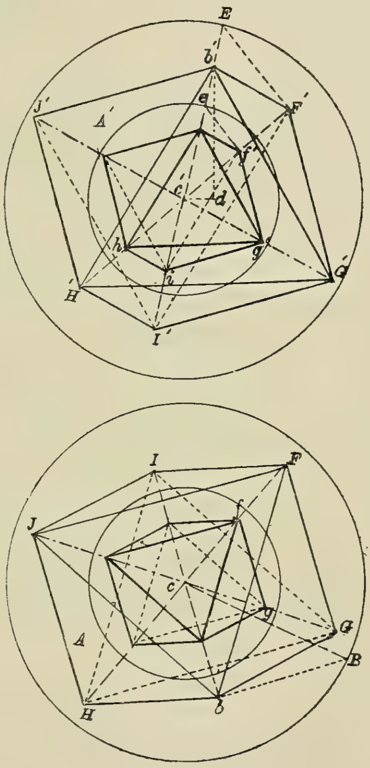


Fig. 270.

ing manner. Draw a line from b on plan perpendicular to bc, and equal in length to the perpendicular height of b' above c', viz. db'. This will give point B. Join Bc, which is the true length required, consequently it is the radius of the circumscribing sphere. With c and c' as centres, draw circles of this radius, which will be the projections of the sphere required.

With c and c' as centres, draw circles half an inch in radius.

These are the projections of the sphere circumscribing the required octahedron.

Join all the angular points of the given solid to c in plan, and c' in elevation.

Produce $c'b'$ till it meets the outer circle in E, cutting the inner circle in e. Join EF′. From e draw a line parallel to EF′, and cutting F′c' in f'. Draw a line from f' parallel to F′G′, till it meets G′c' in g'; also lines from g' parallel to G′I′ and G′H′, till they meet I′c' and H′c' in the points i' and h'. Proceed in the same manner with the remaining points till the elevation of the solid is completed.

To determine the plan. Draw a projector from f' till it meets Fc in f. Commencing at this point, draw lines in succession parallel to the corresponding lines of the given solid, as described for the elevation, and complete the plan.

Problem 221.

1. *From a given point on the surface of a given sphere to project an inscribed cube.*

2. *Inscribe an octahedron in the cube.* Fig. 271.

Let A and A′ be the plan and elevation of the given sphere, and c and c' the projections of the given point.

Find o and o', the centres of A and A′ (Prob. 33, Plane Geometry).

Note.—It is advisable to arrange the cube so that four of its edges are parallel to the V.P. If the given point is in an inconvenient position for this, the positions of the ground line and elevation of the sphere could be altered.

First determine the length of one edge of the cube (Prob. 28).

With o' as centre, and $o'c'$ as radius, draw an arc. With c' as centre, and a radius equal to one edge of the cube, draw an arc intersecting the arc from c' in f'. Draw a line from c' perpendicular to $f'c'$ till it meets XY in J. With J as centre, and radius Jc', draw an arc till it meets XY in C′. Draw a projector from C′ till it meets a horizontal line from c in C.

Next determine the radius of a circle that will inscribe in a square with edges equal to $f'c'$, in the following manner :—Bisect $f'c'$ in k. With k as centre, and radius kc', draw an arc till it meets a perpendicular on k in l. Join lc'. With C as centre, and radius equal to lc', draw an arc intersecting a horizontal line from o in O. With the same radius, and O as centre, draw a

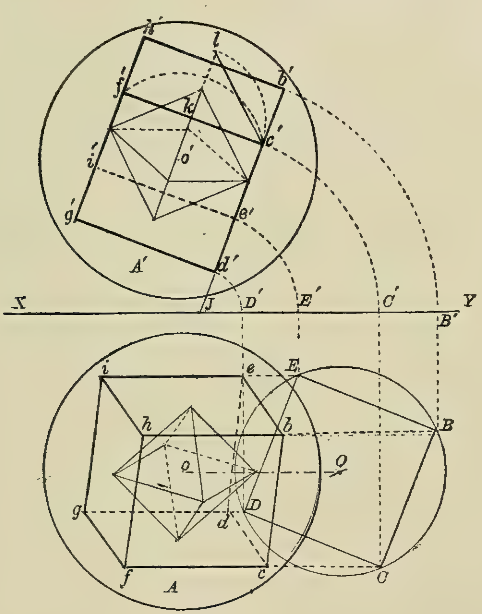


Fig. 271.

circle. Mark off round this circle, commencing at C, with $f'c'$ as distance, B, E, and D, and join them to complete a supplementary elevation of the square. Draw projectors from the corners of this square till they meet XY in the points B′, E′, and D′. With J as centre, and each of these points in succession as radius, draw arcs till they meet the line Jc' produced in the points d', e', and b'. Draw lines from each of these points parallel to and equal in length to $f'c'$, and join their ends by the line $g'h'$. This will complete the elevation.

Draw projectors from each of the points in the elevation till

they meet horizontal lines drawn from corresponding points in the square BCDE, and join them to complete the plan.

2. Find the centre of each side of the cube in plan and elevation by drawing diagonals. Join each of these points, as shown, which give the plan and elevation of the inscribed octahedron.

PROBLEM 222.

1. *To inscribe a given dodecahedron in a sphere.*
2. *To inscribe a sphere in a given dodecahedron.* Fig. 272.

Let A and A′ be the plan and elevation of the given solid.

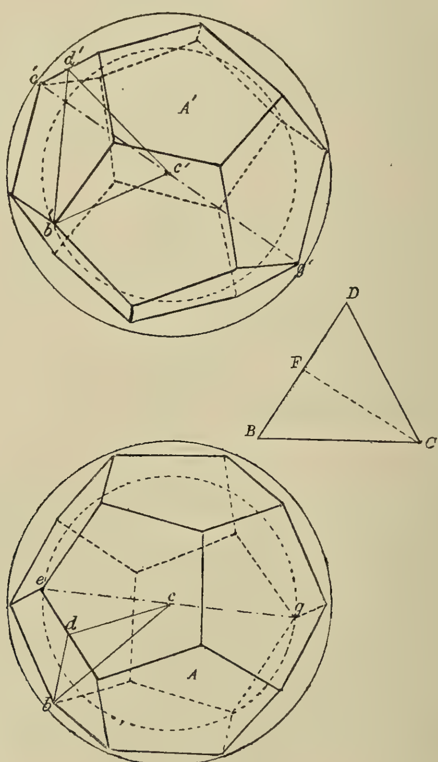


Fig. 272.

Find the centre of the solid by joining any two opposite

points, as *e* and *g*, by a line and bisecting it in *c*. Join any angular point to the centre, as *b*, and bisect the edge on the same face, opposite *b*, in *d*. Join *db* and *dc*. Proceed in the same manner with the elevation, and find the corresponding points *c'*, *b'*, *d'*, and join them.

From the projections of these three lines, forming a triangle, determine their true lengths and angles, as CBD (Prob. 78).

As C represents the centre of the solid, and B one of the angular points, BC must be the radius of the sphere required. Describe circles, with this radius, from *c* and *c'*.

2. The faces of the solid are tangential to an inscribed sphere. BD represents the centre line of one of these faces. Draw a line from C perpendicular to BD, as CF. Then CF is the radius of the inscribed sphere. Draw circles of this radius from *c* and *c'*.

CHAPTER XXXV

INTERPENETRATION OF SOLIDS

SOLIDS mutually intersecting are said to *interpenetrate,* and the lines formed by the intersection of their surfaces are called *lines of interpenetration.*

The principle of this subject is similar to that illustrated in Sections of Solids (Chap. XXX.), that is, where solids are intersected by cutting planes.

When one solid intersects another, its various surfaces may be taken to represent so many cutting planes, and their intersection with the other solid determines the line of interpenetration.

One plane always intersects another in a right line, and any two points in this line will determine its direction. The following problems have been devised to show the various methods used to determine these two points.

When one or both of the solids is composed of curved surfaces, the line of interpenetration is a curve, which necessitates the finding of a succession of points to determine its form.

Note.—The method of projecting the various solids given in this chapter is fully described in Chaps. XVIII. and XX.

PROBLEM 223.

To determine the interpenetration of two rectangular prisms, one to be $1\frac{1}{2}'' \times 1'' \times 1''$, *with its longer edges parallel to the* V.P.

and one of its faces inclined to it at an angle of 30°; *the other prism to be* $2'' \times \frac{3}{4}'' \times \frac{3}{4}''$, *with its longer edges parallel to the* H.P. *and inclined to the* V.P. *at an angle of* 25°, *and one of its faces inclined to the* H.P. *at an angle of* 30°. *The two axes to bisect one another.* Fig. 273.

Draw the plan and elevation of the vertical prism in the

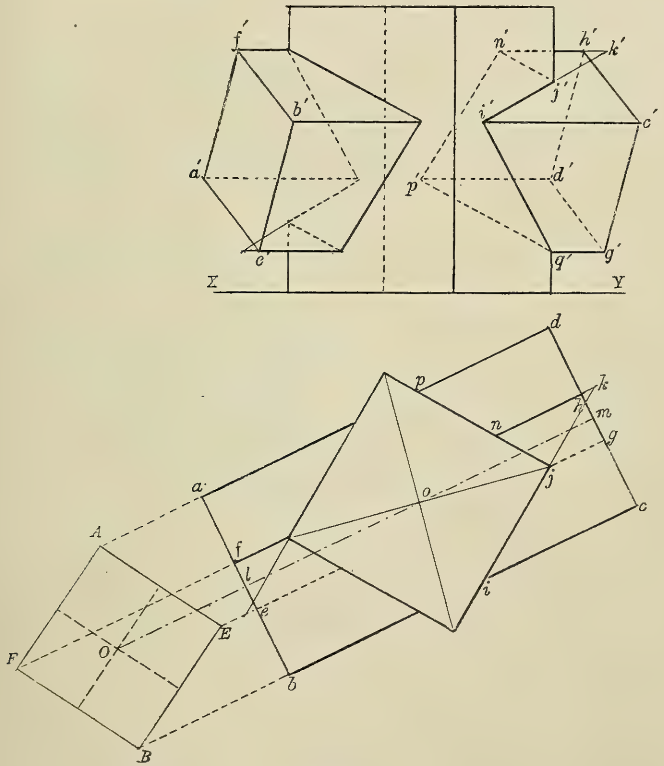


Fig. 273.

position given. Draw diagonals to the plan to find the axis *o*. Draw a line through *o* at an angle of 25° with the V.P., and set off upon it the points *l* and *m*, each one inch from *o*. This is the axis of the horizontal prism. Draw the lines *ab* and *cd* perpendicular to the axis.

Construct a supplementary end elevation of the horizontal prism, using *ab* as a ground line, as follows :—

Produce the axis, and make *l*O equal to half the height of the vertical prism. The point O is the position of the axis in the end elevation. Draw the end elevation AEBF, making one side inclined at an angle of 30° with *ab*. Draw lines from the points A, E, B, F, parallel to the axis, to determine the edges of the horizontal prism on plan.

The positions of the edges of this prism in elevation are determined by making the heights of the points a', e', b', f' above XY equal to the distances of the points A, E, B, F from *ab*. Draw horizontal lines through these points till they meet projectors from the corresponding points on plan, and complete the elevation.

To determine the line of interpenetration, proceed as follows :—

Draw a projector from *i* till it meets the corresponding edge in elevation in i'. Produce *ij* till it meets the edge *fh* produced in *k*. Produce the edge $f'h'$ till it meets a projector from *k* in k'. Join $i'k'$, cutting the edge of the vertical prism in j'.

Draw a projector from *n* till it meets the corresponding edge in elevation in n'. Join $j'n'$.

Draw a projector from *p* till it meets the corresponding edge in elevation in p'. Join $p'n'$ and $p'q'$. This completes the line of interpenetration for one end of the horizontal prism. Proceed in the same manner with the opposite end. All the necessary lines for its construction are shown.

Problem 224.

The projections of a pyramid and a rectangular prism being given, to determine their interpenetration. Fig. 274.

Produce the end of the prism on plan, and assume it to be a ground line, as X′Y′. On this ground line construct a supplementary elevation of the two solids, and make the corners

A, B, C, D of the prism the same height above X′Y′ as the corresponding corners in elevation are above XY.

Draw lines from each of the corners A, B, C, D, parallel to X′Y′, till they meet the edge of the pyramid in the points E, F, G, H. Draw projectors from these points till they meet the

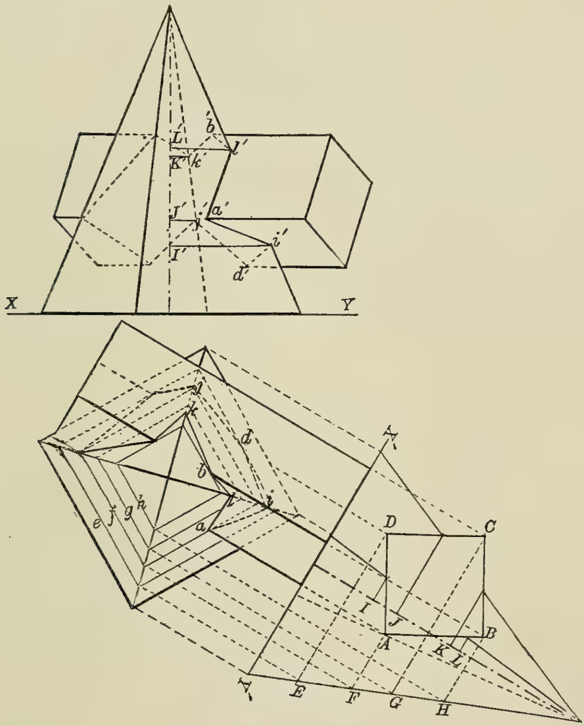


Fig. 274.

corresponding edge of the pyramid on plan. From these four points draw the lines e, f, g, and h to each side of the pyramid parallel to its base.

We now have four contour lines on the pyramid at the same levels as the edges of the prism, and where the edges meet these contours must give the points of interpenetration; *e.g.* the edge A is at the level of the contour F, so where the corre-

sponding edge of the prism on plan meets the contour f gives the point a. The points b and d are determined in the same manner.

We must now determine where the edges of the pyramid are intersected by the prism, as follows :—

Draw lines from the points where the edges of the pyramid, in supplementary elevation, meet the prism parallel to X′Y′ till they meet the axis of the pyramid in the points I, J, K, L. Set up above XY, in elevation, corresponding heights in the points I′, J′, K′, L′. Draw horizontal lines from these points till they meet the edges of the pyramid in the points i', j', k', l'. Join $a'i'$ and $a'l'$, $i'd'$ and $d'j'$, $l'b'$ and $b'k'$. Proceed in the same manner with the opposite end of the prism to complete the elevation.

The plan could be completed by drawing projectors either from the elevation or from the supplementary elevation.

Problem 225.

1. *To determine the interpenetration of a given hexagonal pyramid with a given triangular prism.*
2. *Develop the true shape of the line of interpenetration.* Fig. 275.

1. Draw projectors from the points a, b, and c on plan till they meet the corresponding edges of the pyramid in the points a', b', and c' in elevation, and join them.

Assume a vertical plane touching the edge i of the prism, parallel to lines kl and no of the base of the pyramid. Draw ef through i parallel to kl. Draw a projector from f till it meets the corresponding edge of the pyramid in elevation in f'. Draw $f'e'$ parallel to $k'l'$ touching the edge of the prism in h'. Join $a'h'$. This vertical plane will also contain the line jg, which is on the upper surface of the pyramid immediately above ef.

Draw a projector from g till it meets the corresponding edge of the pyramid in g'. Draw $g'j'$ parallel to $n'o'$, touching the edge of the prism in i'. Join $c'i'$.

Proceed in the same manner to find the points on the back of the prism.

Draw projectors from the points of the pyramid on the top surface of the prism till they meet the corresponding edges of

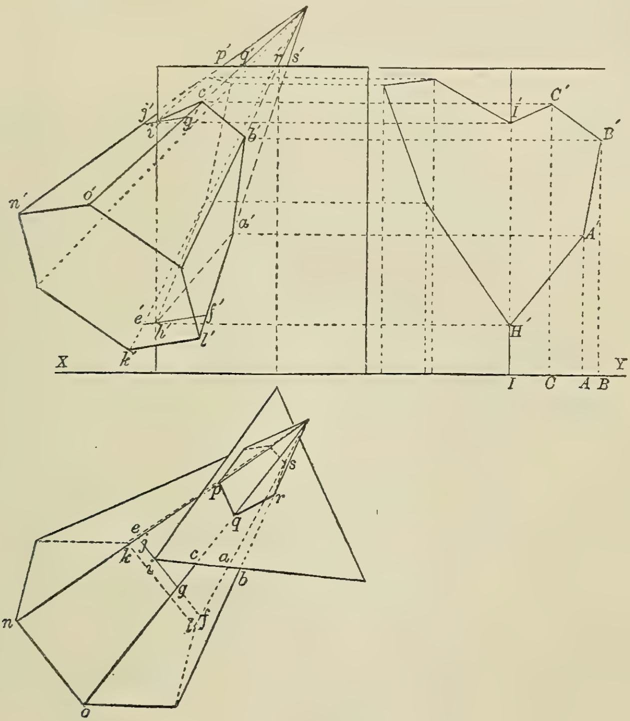


Fig. 275.

the prism on plan, and join them. These give the plan of the interpenetration of the pyramid with the top of the prism.

2. To develop the line of interpenetration. At any convenient point I on XY draw a perpendicular to represent the edge i of the prism.

From I set off on XY the points C, A, and B equal to the distances of c, a, and b on plan, and draw perpendiculars till they meet horizontal lines drawn from the corresponding points in elevation in the points I′, C′, B′, A′, and H, and join them.

Proceed in the same manner with the points at the back of the prism to complete the development.

The plan of the pyramid on the top of the prism, being on a H. plane, is the true shape of its line of interpenetration.

Problem 226.

The projections of a rectangular and a triangular prism being given, to determine the line of interpenetration. Fig. 276.

In the preceding problems of this chapter one of the solids given is resting on the H.P. In this instance both the given solids are inclined to both planes of projection.

First determine the H. traces of each solid, as ABDC and EFG, in the following manner :—

Produce the longer edges of each solid in elevation till they meet XY. Draw projectors from the points on XY till they meet the corresponding edges of the solids produced in plan.

We will now assume a H. plane cutting through the two solids at any convenient level, as X′Y′. Find the H. traces of the two solids at this level by drawing projectors from the points where the edges of the solids intersect X′Y′ till they meet the corresponding edges in plan, as *abdc* and *efg*. Where these traces intersect will determine points in the line of interpenetration, *e.g.* *h* and *k*. To draw the line *ij*. We know that it is determined by the intersection of the side *cd* of the rectangular prism with the side *gf* of the triangular prism; so produce the H. traces of these two sides CD and GF till they meet in H. This is the H. trace of the line *ij*. Draw it towards this point, stopping it against the edges of the two prisms.

Proceed in the same manner with the line *ln* through point *k*. The H.T. of this line is found by producing GE till it meets CD in K. Join *nj*.

The H.T. for the line *io* is found by producing AC till it meets FG produced: this comes beyond the limits of the

drawing. All the H. traces of the remaining lines are found in

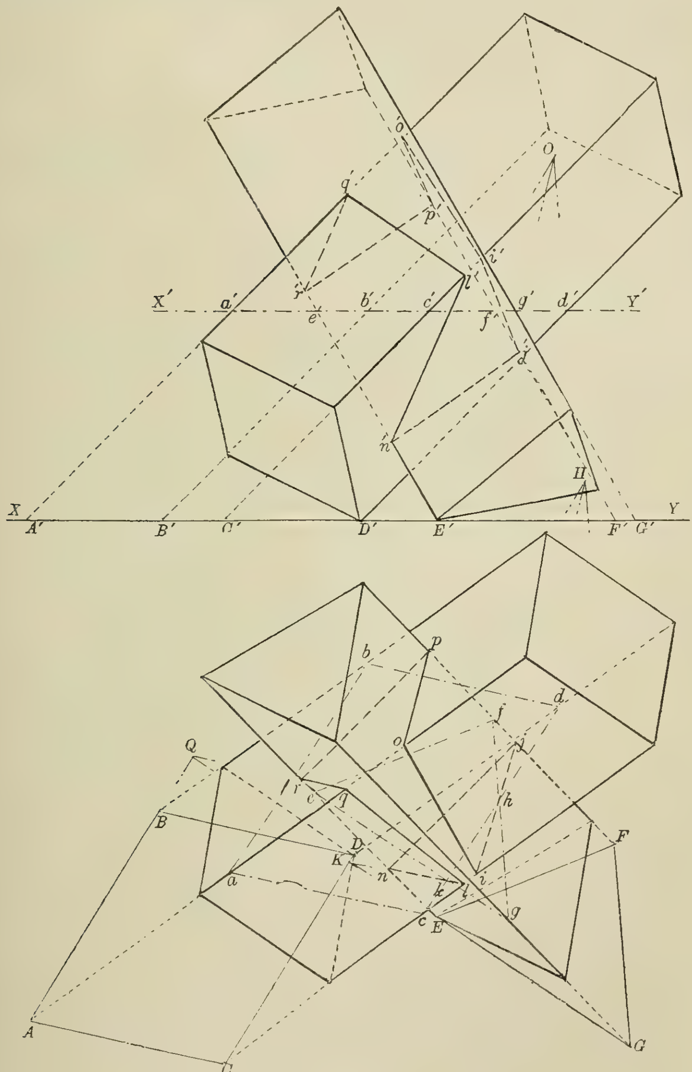


Fig. 276.

the same manner. Q is the H.T. for the line qr. Determine these lines to complete the plan.

To complete the elevation, draw projectors from the points

in plan till they meet the corresponding edges in elevation, and join them.

Problem 227.

The projections of a quadrilateral and a triangular pyramid being given, to determine their interpenetration. Fig. 277.

In this problem we will assume two V. planes cutting through the solids, and determine V. traces of the pyramids upon them.

In any convenient position on plan draw X′Y′ and X″Y″, and assume them to be the H. traces of two V. planes.

Determine the V. traces of the two solids on each of these planes by drawing projectors from the points where their edges cut X′Y′ and X″Y″ till they meet the corresponding edges produced in elevation, as A′D′C′B′ and E′F′G′, and *a′d′c′b′* and *e′f′g′*.

Note.—The traces on the plane X″Y″ are shown as sections in elevation to make them more conspicuous.

Where the V. traces on the same plane intersect is a point in the line of interpenetration; *e.g.* the intersection of A′D′C′B′ with E′F′G′ in *o′* is such a point. As *o′* is common to the sides B′C′ of one pyramid and E′G′ of the other, a second point in the line containing *o′* can be found by producing the corresponding lines of the other V. traces; for instance, by producing the lines *b′c′* and *e′g′* till they meet in O′, which is the V. trace of the line required. Draw a line from O′ through *o′* till it meets the edges of both pyramids.

All the other lines required can be determined by finding their V. traces in the same manner: that is, by producing the V. traces of the sides of the pyramids containing the line till they meet, as described for the H. traces in the preceding problem; *e.g.* the line *p′n′* is contained by the sides C′B′ and E′F′, the traces of these sides meet in N′, which is the V. trace of the line *p′n′*.

The line $n'r'$ is the intersection of the sides A′B′ with the side E′F′. R′ is the V. trace of this line.

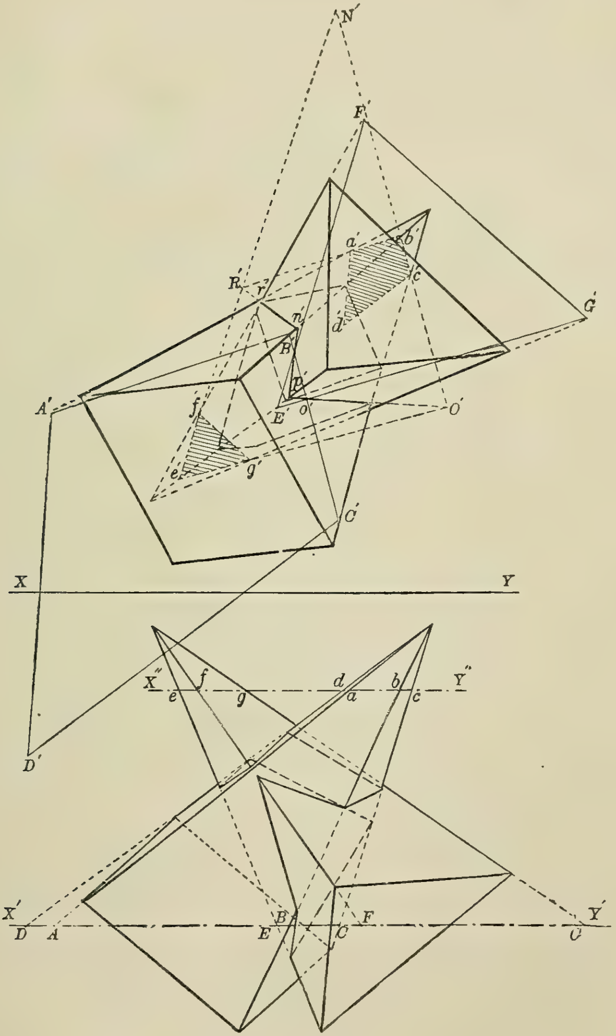


Fig. 277.

The points of the line of interpenetration on plan are determined by drawing projectors from the points in elevation till

they meet the corresponding edges of the solids on plan. Join them to complete the drawing.

PROBLEM 228.

The projections of a group of three solids being given, to determine their interpenetration. Fig. 278.

Let A, B, and C be the plans, and A′, B′, and C′ the elevations of the three given solids.

In this illustration we will take a series of contours to the three solids, *i.e.* a succession of H. traces of them at different levels, in the following manner:—

In any convenient position in the upper part of the elevation draw three horizontal lines: as 1′, 1′; 2′, 2′; and 3′, 3′. These lines represent the V. traces of the planes that represent the different levels of the contours on plan.

The contours are found in precisely the same way as the H. traces of Prob. 226, *i.e.* by drawing projectors from where the V. traces intersect the various edges in elevation till they meet the corresponding edges in plan, and joining them.

All the contours on plan figured 1 are at the level of the V. trace 1′, 1′; those figured 2, at the level of 2′, 2′, etc.

When a solid is situated as B, it is not possible to draw projectors directly from one projection to the other, but they may be drawn as follows:—

Take any point D′ level with a'. Join D′b'. Draw a projector from D′ till it meets a horizontal line drawn from a in D. Join Db. Draw projectors from where the V. traces 1′, 1′ and 2′, 2′ meet D′b' in c' and d' till they meet the line Db in the points c and d. Draw horizontal lines from these points till they meet the line ab in the points 1 and 2.

Note.—All the contours on the same face are parallel to each other.

Having determined all the necessary contours, proceed as follows:—

Where two contours bearing similar figures intersect, determine a point in the line of interpenetration ; *e.g.* 2 and 3 of B

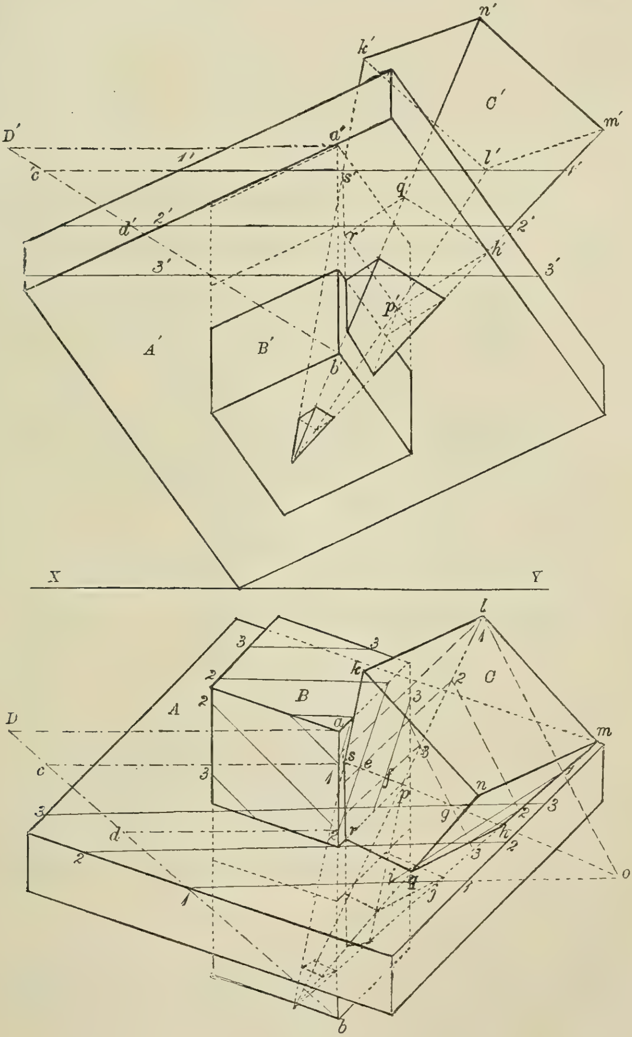


Fig. 278.

intersect 2 and 3 of C in *e* and *f*. Draw a line through these two points from the edge of C in *s* to the intersection of the faces at *p*.

We have next to determine the intersection of the face *lm* of C with the upper surface of A. Produce any two contours of the same number belonging to these two faces till they meet; *e.g.* produce 1, 1 of A and 1, 1 of C till they meet in *o*. Join *po* intersecting the edge of C in *h*. The student will see that the contours 3 and 3 belonging to these two faces also intersect on this line in *g*.

Next determine the intersection of the face *nm* of C with the upper surface of A. Produce any two similar contours belonging to these two faces, as before, till they meet; *e.g.* 1 and 1 produced meet in *i*. Join *h* to *i*, intersecting the edge of C in *q*. Produce 1 of the face *kn* till it meets 1 of A in *j*. Draw a line from *j* through *q* till it meets the solid B in *r*. Join *rs*.

Note.—Only three contours are taken, to avoid confusion of lines, but they could be continued through the lower part of the figure to complete the plan.

The points in the elevation are found, as usual, by drawing projectors from the points in plan till they meet the corresponding edges, and joining them.

Problem 229.

To determine the interpenetration of three given cylinders.
Fig. 279.

Let A, B, and C be the plans, and A′, B′, and C′ the elevations of the three cylinders.

Bisect *ab* of B in *c*. With *c* as centre, and radius *ca*, draw a semicircle and divide it into any number of equal parts (6). Draw lines from these divisions perpendicular to *ab* till they meet the cylinder A in the points *f*, *g*, *h*, etc.

Bisect *e′d′* of B′ in *c′*. Set off from *c′* divisions equal to those on *ab*, and draw horizontal lines till they meet projectors drawn from the points *f*, *g*, *h*, etc., in the points *f′*, *g′*, *h′*, etc. Draw a

fair curve through these points for the line of interpenetration between the cylinders A′ and B′.

Rotate the cylinder C till it is parallel to the V.P., in the following manner:—

With *o* as centre, and radii *n* and *m*, draw arcs till they

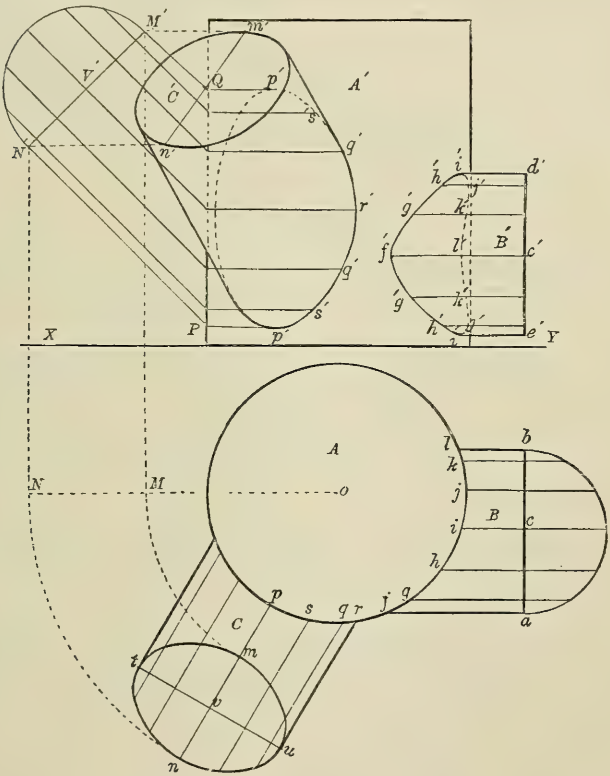


Fig. 279.

meet a line drawn from *o* parallel to the V.P. in N and M. Draw projectors from N and M till they meet horizontal lines drawn from *n*′ and *m*′ in the points N′ and M′. Join N′M′. Draw M′Q and N′P perpendicular to N′M′. This is a supplementary side elevation of the cylinder C.

Bisect the line N′M′ in V′. With V′ as centre, and radius

V′N′, draw a semicircle and divide it into any number of equal parts (6). Draw lines from these divisions perpendicular to N′M′ till they meet the side of the cylinder A′.

Draw the diameter *tu* to the end of the cylinder C, and set off upon it divisions equal to those on N′M′. Draw lines from these divisions perpendicular to *tu* till they meet the cylinder A in the points *p*, *s*, *q*, etc. Draw projectors from these points till they meet horizontal lines drawn from corresponding lines of the supplementary elevation in the points *p′*, *s′*, *q′*, etc.

Draw a fair curve through the points.

Problem 230.

To determine the interpenetration of two given cylinders with the given frustrum of a right cone. Fig. 280.

Let A and A′ be the projections of the frustrum, and B, C, and B′, C′, those of the given cylinders.

Divide the lines *ab* and *d′e′* as described in the preceding problem; also draw parallel lines from the divisions as therein described.

Produce the horizontal lines in the elevation till they meet the axis of the frustrum, and assume them to be the V. traces of H. planes common to both solids.

Assume the parallel lines on the plan B to represent H. traces of V. planes the same distance from the axis of B as the H. planes in the elevation.

The intersection of the V. and H. planes will determine points in the line of interpenetration. Their intersection is determined as follows:—

Draw a succession of contours on plan to represent the different levels of the H. planes. We will take one as an illustration. Draw the diameter *mn* on plan. From where the horizontal line *h′h′* meets the side of the frustrum in H′, draw a projector till it meets the line *mn* in H. With *o* as centre, and radius *o*H, draw an arc till it meets the corresponding parallel

lines on plan in *h*, *h*. The line *cc* shows the intersection of the traces of these planes, constructed on the H.P. To determine the elevation of these points, draw projectors from *h*, *h*, till they meet the corresponding line in elevation in the points *h′*, *h′*.

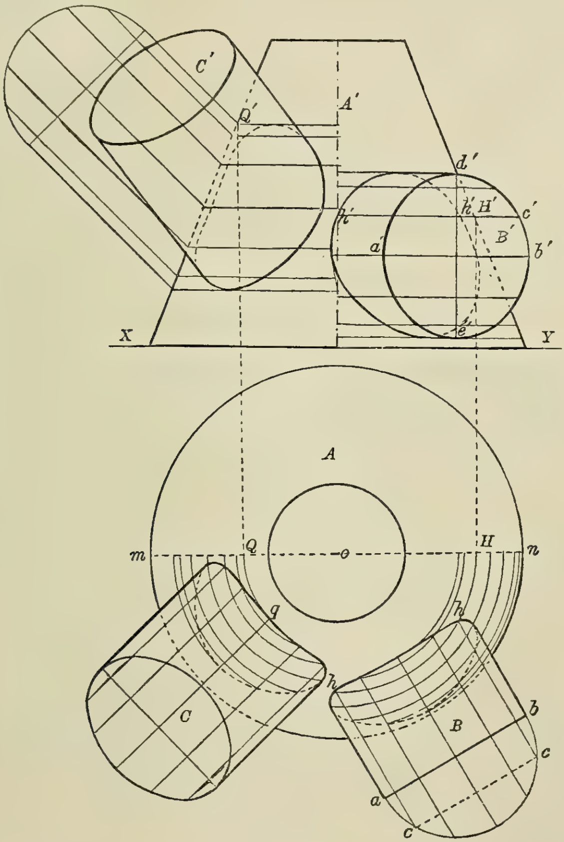


Fig. 280.

Find the remaining points in the same manner, and draw a fair curve through them.

To determine the line of interpenetration between the cylinder C and the frustrum. Find the supplementary elevation of C′, with the divisions and parallel lines, as described in Prob. 229; also draw horizontal lines on the frustrum from the points

where the parallel lines of the supplementary elevation meet it. Let these horizontal lines represent the V. traces of H. planes. Find the contours on plan to represent these levels. Find the points in the line of interpenetration in the same manner as just described for cylinder B, and draw fair curves through them.

Problem 231.

To determine the interpenetration of a given triangular prism with a given sphere. Fig. 281.

Let A, B, and A′, B′, be the projections of the two solids.

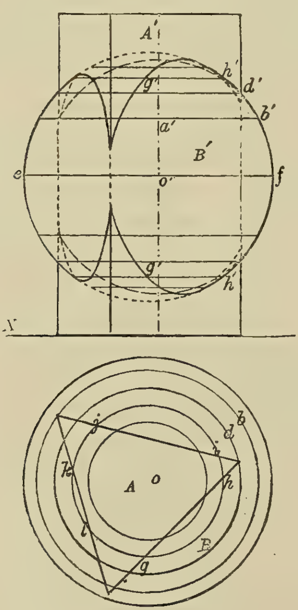


Fig. 281.

Draw a series of horizontal lines to the elevation of the sphere. These lines may be taken at any level, but it is advisable to have them closer together towards the top and bottom of the sphere. The lines in the lower half should also be at the same distance from the diameter ef as those in the upper half of the sphere.

Draw a series of contours to the plan to represent the different levels in the elevation, as follows :—

With o as centre, and radii equal to half the lengths of the horizontal lines of the sphere, in elevation, draw circles : *e.g.* with radius $a'b'$ draw the circle b, etc.

Where these circles meet the plan of the prism, draw projectors till they meet the corresponding lines in the elevation : *e.g.* the circle h meets the prism in the points g, h, i, j, k, and l;

projectors from these points will determine six points on each of the lines h', h'. Determine the points at the other levels in the same manner, and draw fair curves through them.

Problem 232.

1. *To determine the interpenetration of a given sphere with a given right cone.*
2. *To develop the line of interpenetration.* Fig. 282.

Let A, A′ be the projections of the given sphere, and B, B′ those of the given cone.

1. Draw an axis to the cone, also one to the sphere parallel to it.

Draw a succession of horizontal lines in elevation, as a', b', c', etc., and assume them to be the V. traces of horizontal planes. Draw contours on plan to represent these levels, as follows:—

With o as centre, and radii equal to the semi-diameters of the cone at the different levels, draw arcs, as a, b, c, etc.

Find n, the plan of the axis of the sphere.

Contours on the sphere will intersect those at the same level on the cone in the line of interpenetration. We will take one as an illustration. With n as centre, and radius equal to semi-diameter of the sphere at the level of a', draw a circle. This circle cuts the contour of the cone at the level of a in the points e, e, which are two points in the plan of the line of interpenetration. Find the other points of the plan in the same manner.

To determine these points in the elevation, draw projectors from the points in plan till they meet lines at the corresponding levels; *e.g.* draw projectors from e, e till they meet the line at the level of a' in e', e'.

To determine the points g and h, we must make a supplementary elevation of the sphere with its axis at the same distance from the V.P. as that of the axis of the cone. To do

this, take the distance *on*, and set it off from o' on the line $o'n'$. This determines the point N. With this point as centre, draw a supplementary elevation of the sphere, cutting the cone in the points G and H. Find the elevation i' of point i, and join $i'v'$. Draw horizontal lines from the points G and H till they meet

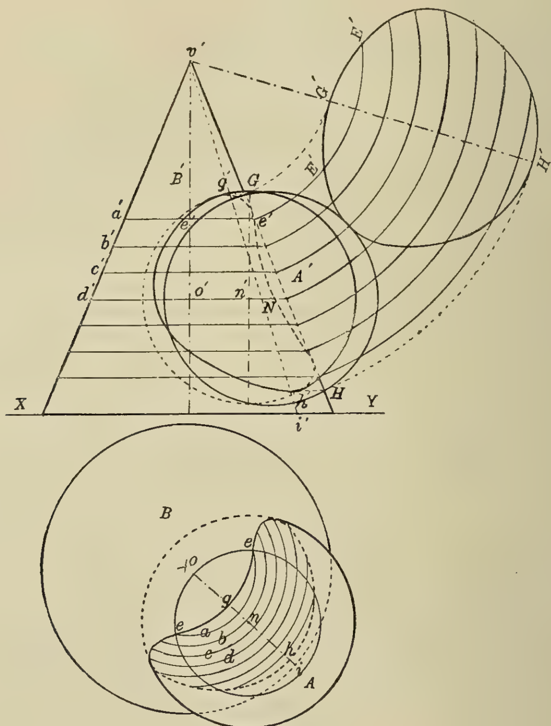


Fig. 282.

the line $i'v'$ in the points g and h'. Draw projectors from these points till they meet the line oi in the points g and h. Having determined all the points, draw fair curves through them.

2. To develop the line of interpenetration. Draw the line v'H′ in any convenient position. With v' as centre, and the points in which the horizontal lines meet the side of the cone as radii, draw arcs as shown. Make each of these arcs equal

to the length of the corresponding contours in plan; *e.g.* make the arc E′E′ equal in length to the arc $e'e'$, half the length to be set off on each side of the line G′H′.

Note.—The best way to measure an arc of this character is to take very small steps with the dividers.

Problem 233.

To determine the line in which two revolving surfaces intersect each other; the surfaces to be the paths of the generatrices of a cone and ellipsoid. Fig. 283.

Let $a'b'$ be the generatrix of the cone, and $f'e'd'$ that of the ellipsoid; also let the axes be inclined to each other, but in a plane parallel to the V.P.

Complete the elevations of the cone and ellipsoid, also project their plans. Produce the axis of the ellipsoid till it meets the axis of the cone.

Assume any number of H. planes to the ellipsoid, and draw their traces.

With the vertex a' as centre, and each of the points in which the H. traces meet the surface of the ellipsoid as radii, as g', h', i', etc., draw arcs till they meet the side of the cone in the points G, H, I, etc. From these points draw lines perpendicular to the axis of the cone till they meet the horizontal lines of the ellipsoid in the points G′, H′, I′, etc. The points G′, H′, I′, etc., must be points in the line of interpenetration between the two surfaces; *e.g.* the point H on the cone moves in a plane of which HH′ is the trace, while the point h' moves in a plane of which $h'n'$ is the trace. The paths of these two points cross each other in two places, H′ being one; the other is directly behind H′, on the back of the ellipsoid. Draw a fair curve through the points G′, H′, I′, etc.

To determine the points in the plan, draw contours to represent the various levels of the H. planes in the elevation.

Note.—It will facilitate the work if the H. traces in the

upper part of the elevation of the ellipsoid are drawn at the same distances from the minor axis as those in the lower half.

Draw a projector from n' till it meets a line drawn through

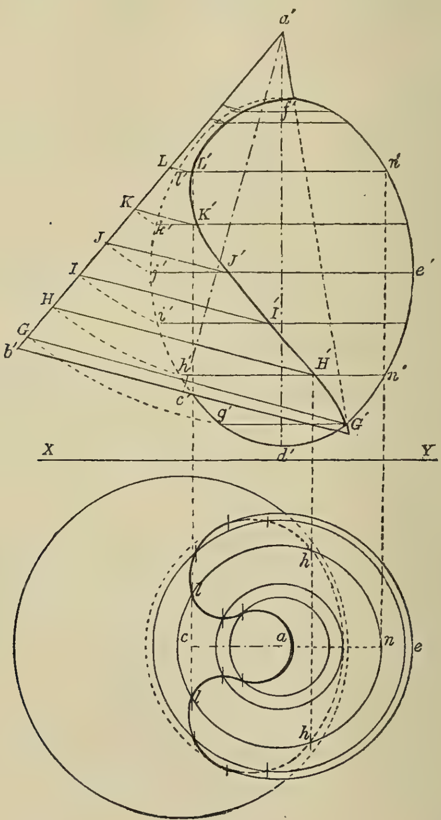


Fig. 283.

ca produced in n. With a as centre, and radius an, draw a circle. This circle is the contour of two of the H. planes: one at the level of n' in the upper part, and the other at n' in the lower part of the ellipsoid. Draw. a projector from H$'$ till it meets this circle in the points h, h, also from L$'$ till it meets it in the points l, l. These are four points in the plan of the

line of interpenetration. Determine the other points in the same way, and draw a fair curve through them.

Problem 234.

To determine the interpenetration of a given frustrum of a cone with its axis parallel to the H.P., *with a given right cone with its axis parallel to the* V.P. Fig. 284.

Let A, A′, be the projections of the cone, and B, B′, those of the frustrum.

Assume a series of H. planes common to both solids. The traces of these planes are shown by the horizontal lines C′, G′, H′ etc., in elevation.

The intersection of the contours of each solid on plan, representing the levels of these H. planes, will determine points in the line of interpenetration.

Note.—If B were a cylinder, instead of the frustrum of a cone, the problem would be comparatively simple, as its contours would then be rectangles, with widths varying in proportion to the level of each contour; but in the case of the frustrum of a cone these contours are hyperbolas.

Construct the contours of B in the following manner:—

Assume a series of V. planes perpendicular to the axis: the H. traces of these planes are shown by parallel lines in plan. Bisect the line *a*C in *c*. With *c* as centre, and radii equal to the semi-diameter of B, on each of the H. traces draw semicircles. Draw the line *c*E perpendicular to *a*C, and divide it in the same manner as the line *c′e′* in elevation. These divisions give the levels of the contours in a supplementary end elevation of the upper part of B. Projectors drawn from the points where these lines intersect the semicircles till they meet the H. traces on B will determine points in the contours; *e.g.* projectors drawn from the nine figures on the line I, parallel to the axis, will determine the nine points bearing corresponding figures in the contour at the level of I: this also represents a similar contour

in the lower part of B′ at the level of I′. Determine the other contours of B in a similar manner.

Note.—It is sometimes necessary to take additional contours to determine the proper curve of the line of interpenetration, as

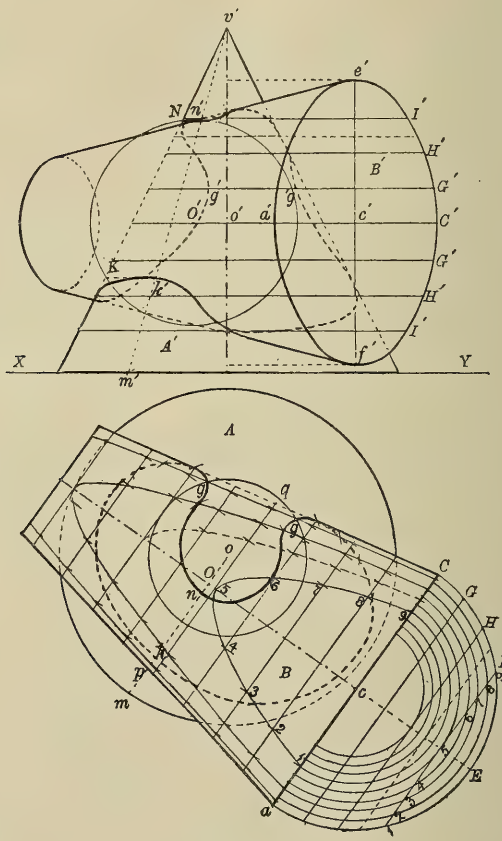


Fig. 284.

shown by the dotted line between the levels of H′ and I′ in the upper part of B′.

We must now determine the contours on the plan of the cone.

With *o* as centre, and radius equal to the semi-diameter of

the upper part of the cone at the level of G′, draw a circle. This will cut the contour at the level of G in *gg*. Determine the other intersections of the contours on plan in a similar manner, and draw a fair curve through the points.

To determine the elevation of the points in the line of interpenetration, draw projectors from the points on plan till they meet the V. traces of the corresponding levels in elevation; *e.g.* draw projectors from the points *g*, *g* on plan till they meet the line at the upper level of G′ in the points *g*′, *g*′. Determine the other points in the same way, and draw a fair curve through them.

To determine the points *n*′, *k*′. Draw the line *om* on plan perpendicular to the axis of B, and cutting it in O. Find the elevation *m*′ of *m*, and join *m*′*v*′. From *o*′ on the line *c*′*o*′ set off *o*′O′ equal to *o*O. With O′ as centre, and radius equal to O*p*, draw a circle. This is a section of B on the line *pq* "constructed" on a plane parallel to the V.P. This section cuts the side of the cone in the points N, K. Draw horizontal lines from these points till they meet the line *v*′*m*′ in the points *n*′, *k*′. Projectors drawn from these points till they meet the line *om* on plan give the points *n*, *k*.

Problem 235.

A point P *is seen from three stations, a, b, and c, all at the same level. The distance ab is* 200 *yards, bc* 125 *yards, and ca* 150 *yards. The altitude of the point at a is* 26°, *at b* 25°, *and at c* 45°. *Determine the plan and elevation of the point. Scale* 1″ = 100 *yards.* Fig. 285.

This point is determined by finding the intersection of the generatrices of three inverted right cones.

Join *ab*, *bc*, and *ca*. Assume *a*, *b*, and *c* to be the vertices of three inverted cones. "Construct" on the H.P. elevations of the generatrices at the given angles; *e.g.* draw a line at *a* at an angle of 26° with *ac*, at *b* 25° with *bc*, and at *c* 45° with *cb*. Draw the axes of the cones *aa*′, *bb*′, and *cc*′ any convenient

height, but all equal. Assume any points d', e' on aa', and set off on the other two axes corresponding points at the same height above b and c.

Draw lines from all these points perpendicular to the axes till they meet the generatrices. These lines represent the levels of H. planes at the same height on each cone. We must now find contours on plan representing these levels, as follows :—

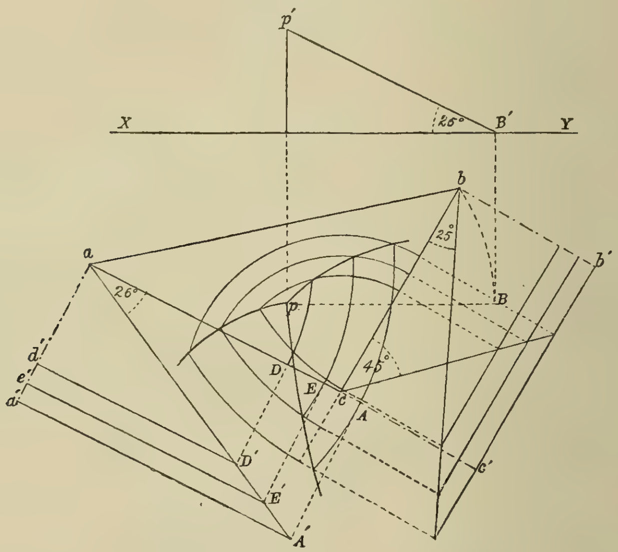


Fig. 285.

Where the lines from a', c', d' on the axis aa' meet the generatrix of a in the points A′, E′, D′ draw lines parallel to the axis till they meet ac produced in A, E, D. With a as centre, and A, E, D as radii, draw arcs. These are the contours on plan of the inverted cone whose vertex is a.

Find the contours to the other cones in the same way. The points on plan in which these contours meet are points in their lines of interpenetration. Draw fair curves through these points till they meet in p, which is the plan of the point required.

To determine the elevation of P. Assume XY to be the

level of the points *a*, *b*, *c*. With *p* as centre, and radius *pb*, draw an arc till it meets a line from *p* parallel to XY in B. Draw the projector BB′. At B′ set off the given angle till it meets a projector from *p* in *p*′. This is the elevation required.

Problem 236.

This problem is an illustration of the interpenetration of solids by horizontal projection, and shows how the intersection of inclined planes in earthworks, roofs, etc., are determined. Fig. 286.

The bases of the solids are resting on the H.P., and the indices to the vertices are their heights above the H.P. in feet.

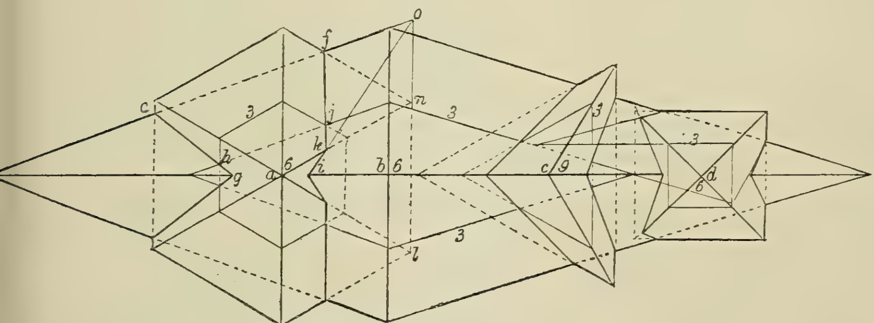


Fig. 286.

Determine a contour common to all the solids, three feet above the H.P., in the following manner :—

As the vertex of the hexagonal pyramid is six feet above the H.P., bisect each of its inclined edges, and draw lines joining these points parallel to its base. Proceed in the same way with the pyramids that have *b* and *d* for their vertices. The vertex *c* of the triangular pyramid is 9 feet above the H.P., so we must trisect its inclined edges to obtain a contour three feet above the H.P.

The intersection of the bases of the solids will determine points in the lines of interpenetration at that level. Where the

contours intersect will determine a second point in each line of interpenetration. *E.g.* we will take the interpenetration of the hexagonal pyramid with the irregular four-sided pyramid. e is a point where their bases intersect, and h is where their contours intersect. Join these points and produce the line till it meets the edge of the pyramid in g.

In the same way join f and j till they meet the edge of the pyramid in k.

To determine the line ki. Produce the bases of the sides containing the line, as ef and ln, till they meet in o. Draw a line from o through k till it meets the edge of the pyramid in i.

Determine the other lines of interpenetration in the same manner. All the working lines are shown.

Problem 237.

To determine the interpenetration of two cylinders with their axes inclined to both planes of projection. Fig. 287.

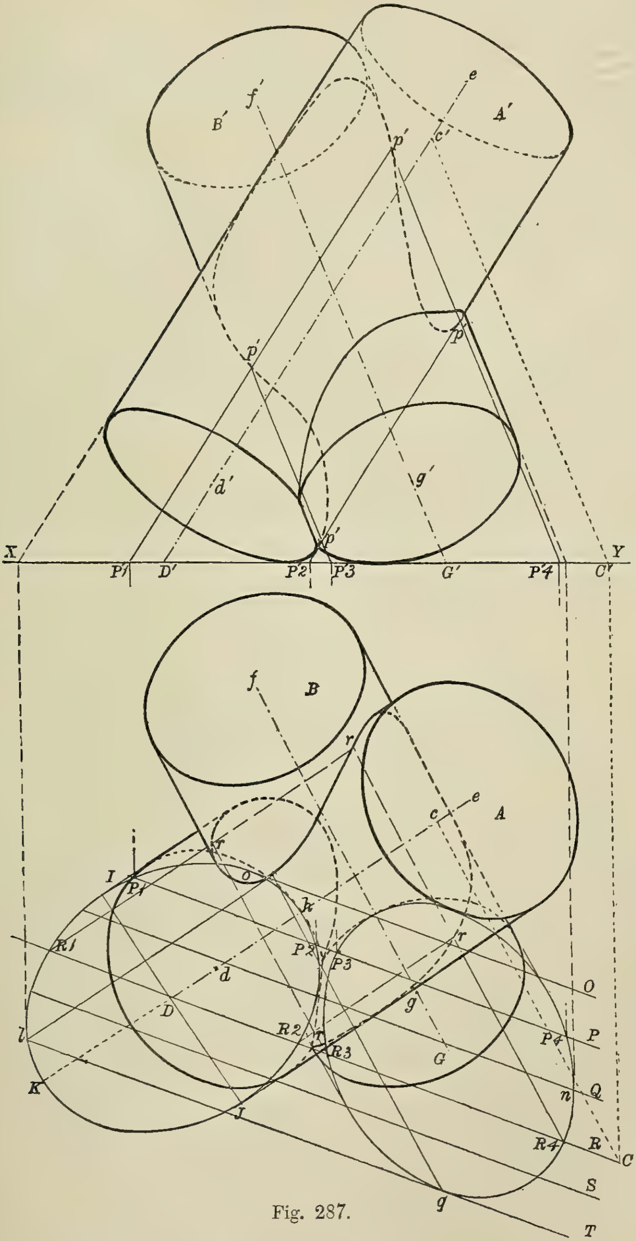
Let A, A′, and B, B′, be the projections of the two cylinders.

Determine the H. traces of each cylinder as follows :—

Draw the two axes in plan and elevation, and produce those in elevation till they meet XY in D′ and G′. Draw projectors from these points till they meet the axes in plan produced in the points D and G. Draw a line through D, perpendicular to the axis eD, till it meets the sides of the cylinder A produced in the points I and J. Set off DK equal to Dh on the line eD produced. Then hK is the major, and IJ the minor axis of the ellipse forming the H. trace of cylinder A. Draw the ellipse (Prob. 181, Plane Geometry). Find the H. trace of cylinder B in the same way.

Determine the H. trace of a plane containing both axes, as follows :—

Take any point c' on the axis e'D′, and from it draw a line



e
f′
A′
B′
c′
p′
p′
p
d′
g′
p′
X
Y
P1′
D′
P2′
P3
G′
P4′
C′
f
B
r
e
c
A
I
P1
r
o
k
R1
P2
P3
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O
D
l
R2
r
R3
P4
P
G
Q
n
K
J
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Fig. 287.

parallel to the axis $f'G'$ till it meets XY in C′. Draw a projector from c' till it meets the axis eD on plan in the point c. From c draw a line parallel to the axis fG till it meets a projector from C′ in C. Draw a line from C through D. This is the H. trace required.

Draw any number of lines parallel to CD, as O, P, Q, R, S, T, and assume them to be the H. traces of parallel planes. As these planes are parallel to the axes of the cylinders, they will intersect their surfaces in parallel lines, and the parallel lines on cylinder A will intersect the corresponding lines on cylinder B in the line of interpenetration; *e.g.* the plane P intersects the H. traces of the cylinders in the points P1, P2, P3, and P4. Draw projectors from these points till they meet XY in the points P′1, P′2, P′3, and P′4. From the points P′1 and P′2 draw lines parallel to the axis $D'e'$ till they meet lines drawn from the points P′3 and P′4 parallel to the axis $f'G'$ in the points p', p', p', p'. These are four points in the line of interpenetration.

Determine the points belonging to the other planes in a similar manner, and draw a fair curve through them.

The points in the line of interpenetration on plan could be determined by drawing lines from the points P1 and P2 parallel to the axis De, and then drawing projectors from the corresponding points in elevation till they meet these lines. Another method is to find the intersection of the corresponding parallel lines on plan. *E.g.* the plane R intersects the H. traces of the cylinders in the points R1, R2, R3, and R4. Draw lines from R1 and R2 parallel to the axis De till they meet lines drawn from the points R3 and R4 parallel to the axis Gf in the points r, r, r, r. Find the points belonging to the other planes in a similar manner, and draw a fair curve through them.

Note.—It is advisable to arrange the H. traces of the planes so as to include the extreme points of the curve; for instance, the points l, o, n, q on plan.

Problem 238.

To determine the interpenetration of a cylinder with a cone, their axes being inclined to both planes of projection. Fig. 288.

Let A, A′ be the projections of the cylinder, and B, B′ those of the cone.

Determine the H. trace of the cylinder, as described in the preceding problem: the H. trace of the cone can be determined from Prob. 216.

The principle of this problem is somewhat similar to the preceding one; but the planes, instead of being parallel, must all pass through the vertex of the cone. The H. traces of these planes will pass through the H.T. of a line drawn from the vertex of the cone parallel to the axis of the cylinder. Determine the H.T. of this line as follows:—

From f' draw a line parallel to the axis of the cylinder till it meets XY. From this point draw a projector till it meets a line from f drawn parallel to the axis D*e*. This point is beyond the limits of the drawing.

Draw any number of lines converging towards this point. These lines represent the H. traces of planes parallel to the axis of the cylinder and passing through the vertex of the cone; consequently they will intersect the cylinder in lines parallel to its axis, and the cone in lines passing through its vertex. The intersection of corresponding lines on each solid will determine points in the line of interpenetration.

We will take one plane as an illustration.

The plane P cuts the H. traces of the two solids in the points P1, P2, P3, P4. Projectors drawn from these points meet XY in the points P′1, P′2, P′3, P′4. Draw lines from the points P′1, P′2 parallel to the axis D′*e*′ till they meet lines drawn from P′3, P′4 to the vertex of the cone in the points p', p', p', p'.

Determine the points belonging to the other planes in the same manner, and draw a fair curve through them.

The points on plan are found by drawing lines from P1, P2

parallel to the axis D*e* till they meet lines drawn from P3, P4

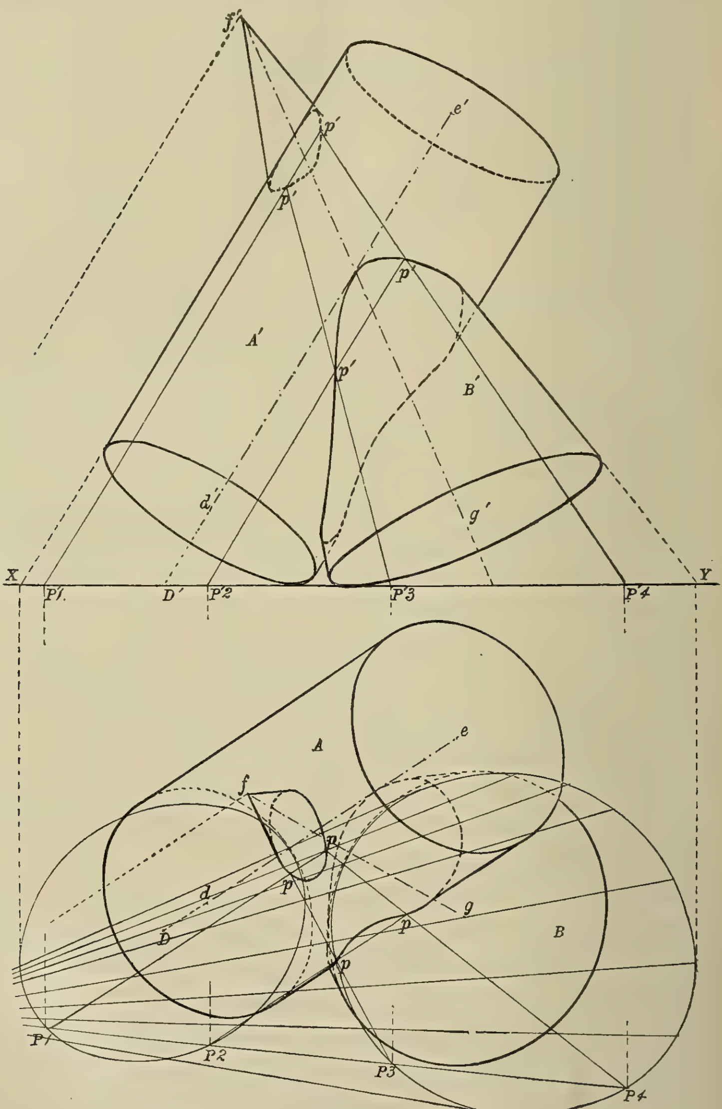

Fig. 288.

drawn to the vertex *f* in the points *p*, *p*, *p*, *p*. Complete the plan.

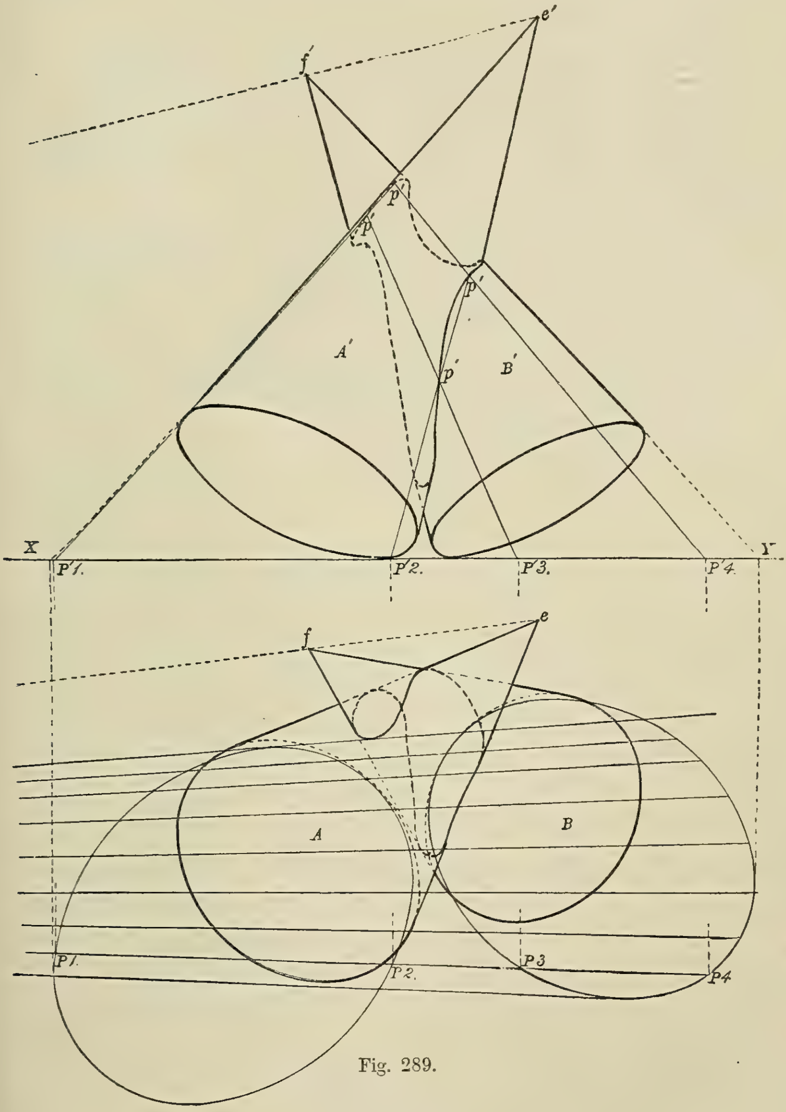

Fig. 289.

Problem 239.

To determine the interpenetration of two cones, their axes being inclined to both planes of projection. Fig. 289.

Let A, A′, and B, B′, be the projections of the two cones.

Determine their H. traces (Prob. 216).

Join the vertices ef, $e'f'$, and determine the H.T. of this line (Prob. 67). This point is beyond the limits of the drawing.

Draw any number of lines converging towards this point, and assume them to be the H. traces of planes passing through the vertices of the cones.

The intersection of these planes with the cones will be in lines passing through their vertices, and the intersection of corresponding lines on each cone will determine points in the line of interpenetration. The lines in connection with plane P are shown on drawing. Determine the other points in the same manner, and complete the curves.

CHAPTER XXXVI

ISOMETRIC PROJECTION

This system of projection was discovered by Professor Farish, of Cambridge, about the year 1820.

In orthographic projection two views of an object are required (plan and elevation), but Isometric projection combines both views in one projection. It possesses also, to a certain extent, the natural appearance of perspective projection, with the additional advantage of uniformity of scale; for this reason it has been termed "the perspective of the workshop."

Its principle is based on the projection of a cube.

In Fig. 290 we have the plan and elevation of a cube with its diagonal $a'b'$ perpendicular to the H.P.

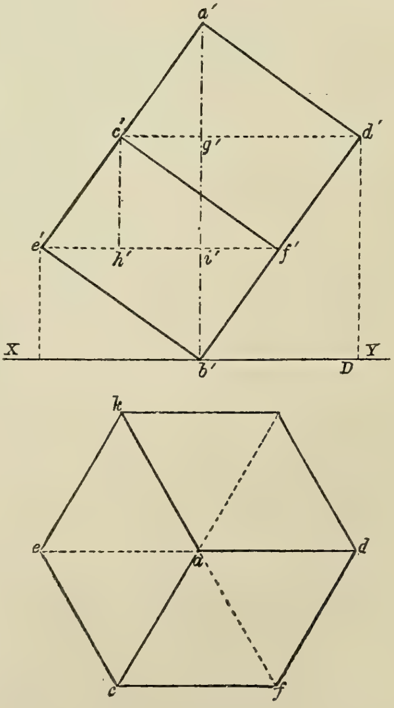

Fig. 290.

Assume $a'g'$ to be the axis of a right cone, and $a'c'$, $a'd'$ its generatrices. As the edges of a cube are all of equal length, the line $a'c'$ must be equal to $a'd'$. On plan the generatrices of a right cone are always drawn of equal length, so ac is equal to ad.

The same thing would apply if we took $c'h'$ for the axis of a cone, or $b'i'$ for the axis of an inverted cone. As all three axes $a'g'$, $c'h'$, and $b'i'$ are equal in length, their generatrices are also equal; therefore the projection on plan of every edge of the cube must be of equal length.

The real length of an edge of the cube is $a'd'$, but the length of its projection is only $g'd'$. Now $a'd' : g'd' :: a'b' : b'd'$, as they are similar triangles.

As $a'b'$ is the diagonal of a cube, and $b'd'$ a diagonal of one of its faces, $a'b' : b'd' :: \sqrt{3} : \sqrt{2}$ (Prob. 123, Plane Geometry).

This is the proportion between the actual scale of an object and its isometric projection.

Problem 240.

To construct an isometric scale: the representative fraction (R.F.) *to be* $\frac{1}{120}$, i.e. $\frac{1}{10}'' = 1$ *foot.* Fig. 291.

Draw a line ab of any convenient length, and at the end erect a perpendicular bc equal to it. Join ac. From b set off bd equal to ac, and join dc. Then $dc : db :: \sqrt{3} : \sqrt{2}$.

Measure off from d on dc a plain scale of $\frac{1}{10}'' = 1$ foot, and from the different divisions draw lines perpendicular to db. Then db is the isometric scale required.

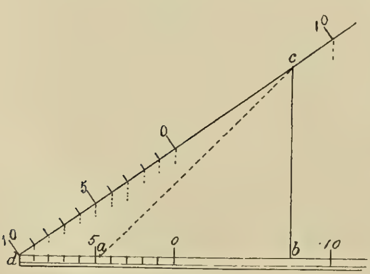

Fig. 291.

Fig. 292 represents the plan of the cube in Fig. 290, "rotated" till the corner d is the lowest point on the drawing. It repre-

sents the isometric drawing of a cube. All the edges, with the exception of the three perpendicular lines, are at an angle of 30° with a horizontal line.

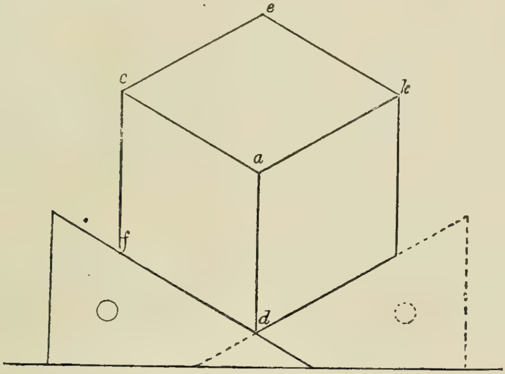

Fig. 292.

A 30° set-square, working against the edge of a tee-square will give all the lines in the projection.

In an isometric drawing we should always commence with the point *a*, called the *regulating point:* the three lines radiating from *a*, as *ac*, *ad*, and *ak* are called the *isometric axes.*

Problem 241.

To determine the isometric projection of a rectangular solid $12'' \times 7\frac{1}{2}'' \times 4\frac{3}{4}''$: *the* R.F. *to be* $\frac{1}{8}$. Fig. 293.

First construct an isometric scale (Prob. 240). The divisions, showing inches, on the scale of real length will each be $\frac{1}{8}''$.

Draw the line *ab* perpendicular to the tee-square, $7\frac{1}{2}''$ long. Draw *ac* and *ad* with the 30° set-square. Make *ac* $4\frac{3}{4}''$, and *ad* $12''$ in length; all the dimensions to be taken from the isometric scale. Draw

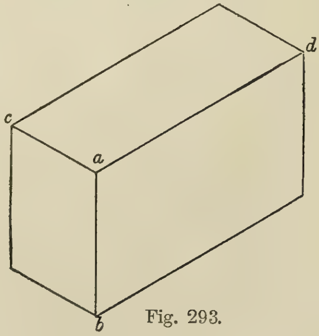

Fig. 293.

lines from *c* and *d* perpendicular to the tee-square. Draw lines from *c* and *b* parallel to *ad*, and from *b* and *d* parallel to *ac*, with the 30° set-square.

Problem 242.

To draw the isometric projection of a circle. Fig. 294.

Let *abdc* be a circle drawn geometrically. Inscribe it in a square *efgh*. Draw diagonals to the square, also bisect its sides, and join the opposite points.

If we refer to Fig. 292, in which *acek* is a square drawn isometrically, we shall see that a diagonal joining *ck* remains its

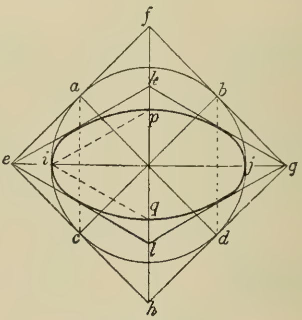

Fig. 294.

true length. The diameter of the circle, *ij*, in Fig. 294 also remains unchanged, for the same reason.

Draw lines from *e* and *g* at an angle of 30° till they meet in the points *k* and *l*. This gives the isometric projection of the square. Draw lines from *i*, at an angle of 30°, to determine the points *p* and *q*. Then *ij* and *pq* are the major and minor axes of an ellipse. Where the lines *ac* and *bd* cut the sides of the isometric square four more points in the ellipse are determined.

In practical work the isometric scale is generally dispensed with, and the dimensions set off at once on the drawing. This method greatly simplifies the work. The relative proportions of the different parts of the object represented are not affected by

this means, but theoretically it represents a larger object than is shown by direct measurement.

The following illustrations in this chapter represent objects drawn in this manner:—

Fig. 295 is an illustration of a double floor, drawn to a scale $\frac{1}{2}'' = 1$ foot, and shows the application of isometric projection to practical construction. It is specially applicable to this class of work, as it enables us to show not only the manner in which

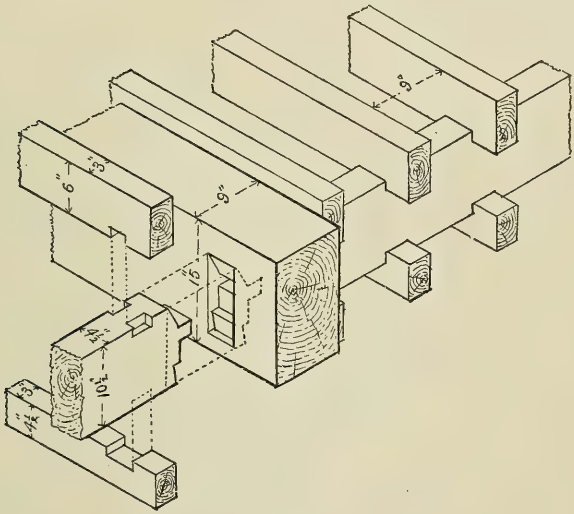

Fig. 295.

the different timbers are framed together when in their proper position, but also the detailed character of each joint.

Fig. 296 represents a hollow cylinder passing through a rectangular solid and projecting an equal length on each side of it.

This may be drawn to any scale.

Draw the rectangular solid, bisect each of the front edges, and join the opposite points; also draw diagonals to its face.

With *c* as centre, draw a quadrant *aed* geometrically, with a radius equal to that of the outer circle of the cylinder, and enclose it in a square, as shown. Draw a diagonal to the square

from *c*, cutting the quadrant in *e*. Draw *cb* parallel to *cd*. Set off from c', the centre of the face, $c'a'$ and $c'h'$ equal to *ca*. From a' set off $a'b'$ equal to *ab*, and draw a perpendicular line till it meets the diagonals in e' and e'. Determine similar points on the opposite side of the circle, and draw a fair curve through them.

Let $h'l'$ represent the projection of the cylinder from the

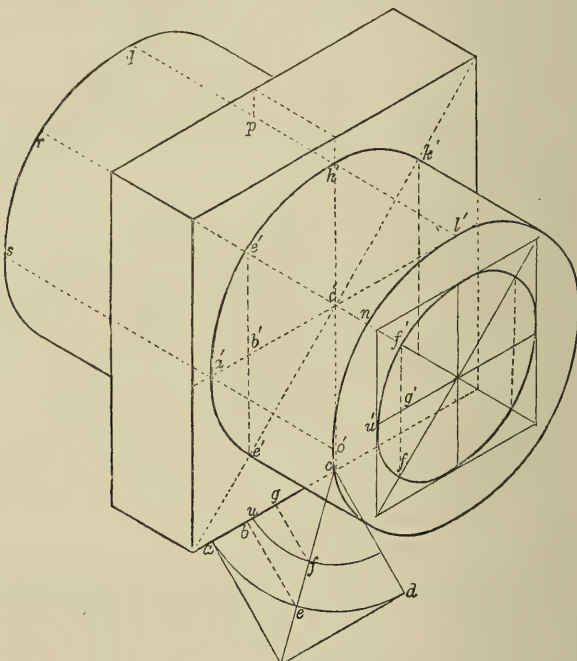

Fig. 296.

face of the rectangular solid. From each of the points in the circle draw lines parallel to $h'l'$ and equal to it in length, as $e'n'$, $a'o'$, etc., and draw a fair curve through the ends of these lines.

Produce $h'l'$, and make $h'p$ equal to the thickness of the block. From each of the points h', e', a', etc., set off on the respective lines produced a distance equal to pl', and draw a curve fair through the points.

With *c* as centre, draw the quadrant *uf* equal to the radius of the bore of the cylinder. Enclose the quadrant in a square, and treat it in the same manner as the larger quadrant. Draw a square on the end of the cylinder with sides equal to twice *cu*, as shown, with diagonals, etc., and find the necessary points for drawing the circle as described for the exterior of the cylinder, and join them with a fair curve.

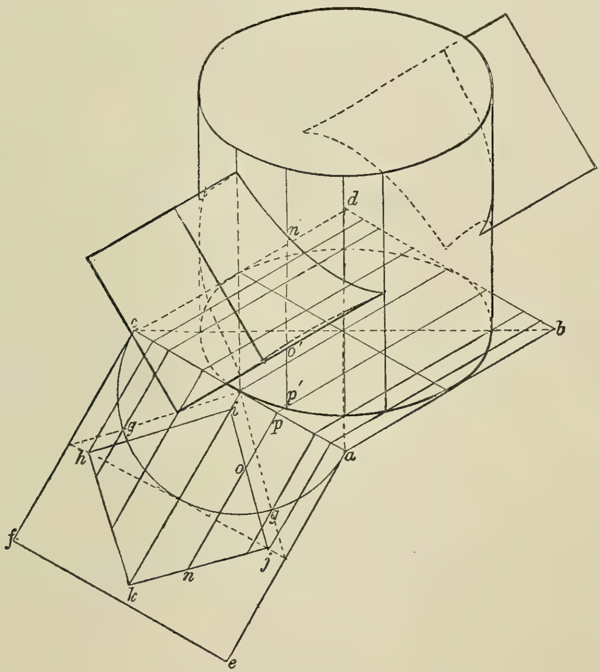

Fig. 297.

Fig. 297 shows the interpenetration of a cylinder and rectangular prism.

First draw the projection of the square *abdc* with sides equal to the diameter of the cylinder. Draw diagonals and diameters to this square.

On the side *ca* draw an end elevation of the cylinder and prism geometrically. Draw a semicircle on the side *ca*, and

obtain the points *g*, *g* where the circle cuts the diagonals. Having obtained corresponding points on the diagonals of the square, draw the base of the cylinder.

On each of the eight points determined for drawing the base, erect perpendiculars equal to *ac* in height, and draw a fair curve through the top points of these lines for the top of the cylinder.

To determine the interpenetration of the prism. Draw any number of parallel lines perpendicular to *ca*, and project them isometrically on the square *abdc*. Where these lines meet the base of the cylinder, draw perpendicular lines, and set off on them corresponding heights to those on the supplementary elevation; *e.g.* on the line erected on p' set off from p', o' and n', equal to the distances of o and n from p—there are four lines on the cylinder with heights corresponding to these. Treat the other perpendiculars in the same manner, and draw a fair curve through the points thus found.

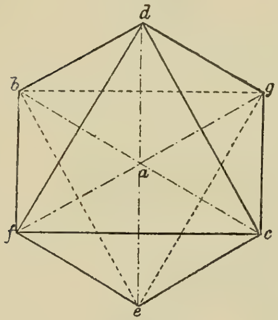

Fig. 298.

Fig. 298 is the projection of an octahedron.

Note.—The isometric axes consist of three lines perpendicular to each other, which is not the case with the corner of this solid, but its axes fulfil these conditions.

Assume *a* as the regulating point, and draw the three axes *bc*, *de*, and *fg* through it at the isometric angles. Make each of the ends of these lines equidistant from *a*, and join them as shown.

In all the foregoing illustrations in this chapter the upper sides of the objects are shown. It is sometimes necessary to show the under side of an object. This can be accomplished by rotating the plan of the cube (Fig. 290) till the point *e* becomes the lowest point in the drawing.

Fig. 299 is an object projected in this manner.

To determine the octagonal surface. Draw a semicircle on the line *ab*, and enclose it in half an octagon geometrically.

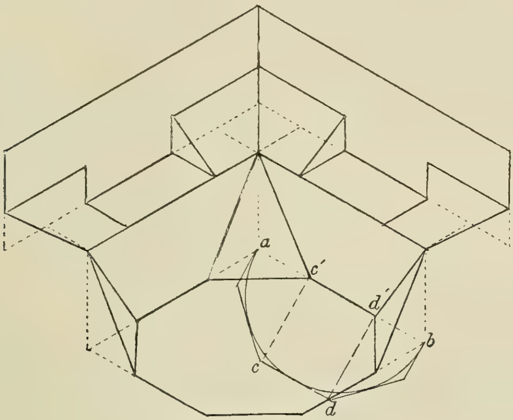

Fig. 299.

Draw lines from the points *c* and *d* perpendicular to *ab* to obtain the points *c′*, *d′*. Set off these distances on each side of the square, and join them as shown.

CHAPTER XXXVII

PRINCIPLES OF MAP PROJECTION

The object of a *map* is to show the relative position of the physical features of a large extent of the earth's surface, such as the coast-lines, rivers, mountains, cities, etc., so that they can be measured by scale.

When the sea is delineated, with its shores, etc., it is called a *chart.*

If the portion represented is small in extent, so that the amount of the earth's curvature is imperceptible, it is called a *plan, i.e.* a plane surface.

So we should speak of a *map* of the British Isles, a *chart* of the German Ocean or the English Channel, or a *plan* of an estate or a building.

As a map represents a curved surface different methods have been devised to express this surface on a plane.

The shape of the earth is nearly spherical,—an oblate spheroid,—and to locate positions, imaginary lines are assumed on its surface ; *e.g.* a great circle, in a plane perpendicular to its axis, is called the *equator.*

From the equator to each pole of the axis is 90°, and at each degree a circle, parallel to the equator, is supposed to encircle the earth : these are called *parallels of latitude*, and are figured from 0° (the equator) to 90° at each pole, and are distinguished by north or south latitude.

The circumference of the equator contains 360°, and through each degree a great circle, in a plane perpendicular to the equator, is assumed to pass round the earth. These great circles intersect each other at the poles, and are called *meridians of longitude;* they are figured from 0° to 180° east and west. The meridian 0° (in England) is supposed to pass through Greenwich. The same meridian on the opposite side of the earth is 180° E. and W.

There are three methods used for projecting these lines on spheres, viz. *Orthographic*, *Stereographic*, and *Gnomonic*.

The plane on which the projection is to be made is called the *primitive*, and the circumference of the sphere drawn to the scale of the projection is the *circumference of the primitive*. Lines drawn from different parts of the parallels and meridians to the eye form the projection.

If the eye is assumed to be at an indefinite distance, or if it is supposed to be directly opposite each point to be projected, the lines of vision are parallel to each other and perpendicular to the primitive. This is called *orthographic projection*.

In *stereographic projection* the primitive is supposed to pass through the centre of the sphere, and the eye to be on a line perpendicular to it, on the surface of the sphere.

In *gnomonic projection* the primitive is supposed to be a tangent plane to the sphere, and the eye placed at the centre of the sphere.

Note.—The circumference of the primitive in gnomonic projection is not the projection of the circumference of the sphere, but of one of the parallels of latitude.

A projection can be either *equatorial*, *meridional*, or *horizontal*, according to the position of the primitive coinciding with, or being parallel to, the equator, meridian, or horizon of the place projected.

In all these projections there is an absence of uniformity of scale: the comparative size of any portion of a continent, etc., can only be ascertained by its position with reference to the

parallels and meridians. Gnomonic projection enlarges the scale of parts at a distance from the centre, while orthographic projection reduces the scale of those parts; consequently they are only used to show the general arrangement of geographical features on hemispheres or very large portions of the earth's surface.

Problem 243.

To determine parallels of latitude and meridians of longitude on a hemisphere by orthographic projection. Fig. 300.

Let WNES be the primitive. Draw WE and NS at right angles to each other.

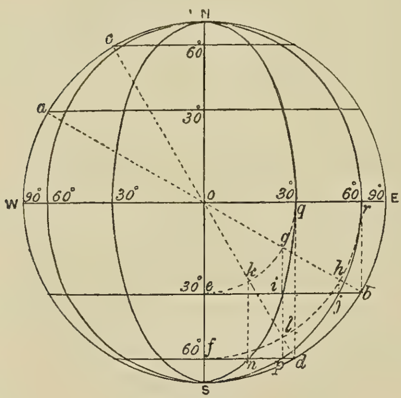

Fig. 300.

Let N and S represent the poles, and WE the projection of the equator.

Draw the line *ab* through the centre of the sphere, at an angle of 30° with the equator, meeting the primitive in *a* and *b*. Draw lines from *a* and *b* parallel to the equator. Then the line at *a* is the projection of the parallel of 30° N. latitude, and that at *b* 30° S. latitude.

Determine the parallels for 60° N. and S. by drawing the line *cd* through the centre, at an angle of 60° with the equator. All the other parallels could be determined in the same way.

To determine the meridians of longitude.

Assume the quadrant ES to be a plan of one-quarter of the sphere, and the lines from S, *d*, *b*, and E to the centre, to be the plan of meridians. From the centre of the sphere, with *f* as radius, draw a quadrant cutting the meridians *d* and *b* in the points *l* and *h*, and the equator in *r*. Draw lines from *l* and *h* parallel to the axis till they meet the parallel at *b* in the points

i and *j*. Set off similar distances on each side of the axis on the parallel at *a*, also on the opposite side of the axis on the parallel at *b*. Draw the quadrant *eg*, and determine points on the parallels at *c* and *d* in a similar manner. Draw fair curves through these points, as shown, for the projection of the meridians. All the other meridians are determined in the same way.

Spheres projected orthographically in different positions are illustrated in Chap. XXI.

Problem 244.

To determine the stereographic projection of a hemisphere, with its parallels and meridians. Fig. 301.

Let WNES be the primitive.

Draw WE and NS at right angles to each other.

Note.—The stereographic projection generally used for this purpose is of a conventional nature: every parallel and meridian is divided into the same number of equal parts.

Divide each diameter and semicircle into six equal parts. These divisions will determine three points in each of the arcs forming the parallels and meridians. *E.g.* *a*, *b*, *c* are the three points in the parallel of 30° N. latitude; and *e*, *f*, *h* in the parallel of 60°. N, *g*, S are the three points of the meridian of 30° E. longitude, and N*d*S those of the meridian of 60°.

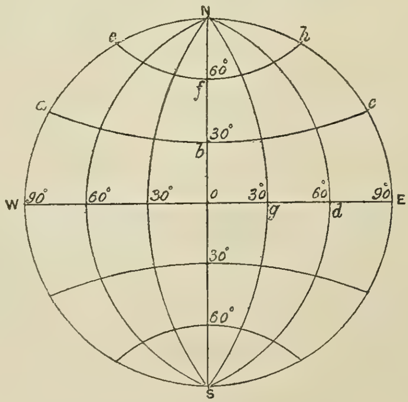

Fig. 301.

From each of these three points we can determine the arc (Prob. 34, Plane Geometry).

Problem 245.

To determine the projection of a portion of a sphere by gnomonic projection. Fig. 302.

Note.—This kind of projection is generally used for the projection of a sphere in the vicinity of its poles, when it is known as *polar projection.*

In polar projection the circle of the latitude for 45° is generally taken as the circumference of the primitive, as *abcd.*

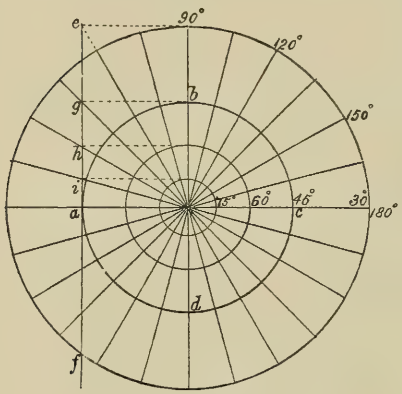

Fig. 302.

To determine the parallels of latitude. At *a* draw a tangent *ef* to the circumference of the primitive, and let it represent the primitive in section, *ac* being the axis, and *a* the pole. The centre of sphere is the position of the eye. A line from the centre at an angle of 75° with *bd* will meet this tangent in *i*; *ai* is the radius for drawing the parallel of 75°. The other parallels are determined in the same manner, the primitive being the parallel for 45°. The distance between each succeeding parallel will increase, the equator being at infinity. The lines radiating from the pole represent meridians 15° apart.

Another form of projection used for maps is that known as "development." In Probs. 51 and 52 the development of a cylinder and cone is illustrated. The curved surface in each instance is developed, or straightened out, into a plane. This is not possible with a sphere.

Fig. 303 is an illustration of this method applied to a map

of the world, and is called *Mercator's projection.* This is based on the development of a cylinder.

The meridians are drawn parallel and equidistant. The parallels of latitude are drawn at right angles to the meridians, and their distances apart increase in proportion to their distance from the equator; so that the degrees of latitude are in propor-

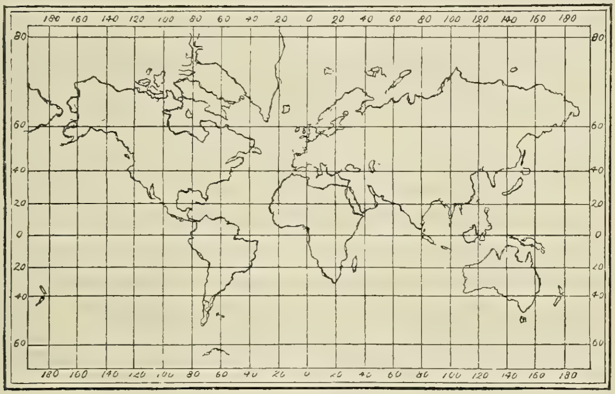

Fig. 303.

tion to the increased size of the degrees of longitude, caused by the meridians being drawn parallel to each other instead of meeting at the poles.

The degrees of latitude in Mercator's projection are set off from tables of meridional arcs; but they can be approximately determined by construction, as shown in the following problem.

PROBLEM 246.

To determine the parallels of latitude for Mercator's projection, by construction. Scale $\frac{5}{8}'' = 20^\circ$ of longitude. Fig. 304.

Note.—As there are 360° in a circle, which is $2\pi \times$ radius, the radius will be $57{\cdot}3^\circ$.

Draw the line $CA = 57{\cdot}3^\circ$. With C as centre, and radius CA, draw a quadrant till it meets a perpendicular on C in B.

Divide this quadrant into nine equal parts, and figure them from 0° to 90°.

Produce the arc of the quadrant from A to E, and make AE = 20°. Join EC.

Assume AC and EC to be the plan of two meridians 20° apart, and determine the plan of the arcs of the different latitudes between them; *e.g.* draw a line from 70° perpendicular to CA

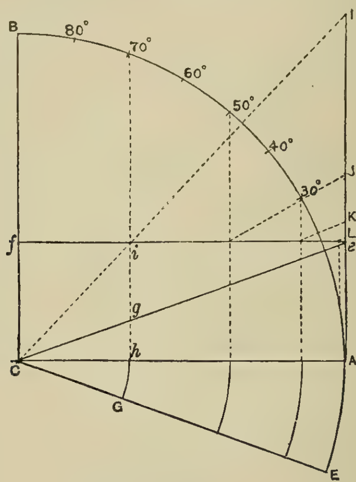

Fig. 304.

and meeting it in *h*. From the centre C draw the arc *h*G. This is the plan of the arc of 70° latitude.

At A draw a perpendicular to CA, and make A*e* equal to the arc AE (Prob. 192, Plane Geometry). Draw *ef* parallel to AC. These two lines will represent the distance of 20° apart of the meridians of longitude. Join *e*C.

As an example, let us determine the distance between the parallels of 60° and 80°. The mean distance between these two will be 70°. A perpendicular from 70° cuts the line *e*C in *g*, and the line *fe* in *i*.

Draw a line from C, through *i*, till it meets A*e* produced, in I. Then AI is the distance between the parallels 60° and 80°.

The other parallels shown are determined in the same manner. AL = the distance from 0° to 20°; AK = 20° to 40°; AJ = 40° to 60°. The construction is shown in the diagram.

To determine the latitude for every 10°, proceed as follows:— For instance, to determine the latitude for 30°, bisect A*e* to find a point to represent 10°. Draw a perpendicular from 20° on the quadrant till it meets the line *fe*, then draw a line from C through this point till it meets AI. The distance from A to this point on AI will be the latitude between 10° and 30°. A perpendicular from 40° on the quadrant, served in the same way, will give on AI the latitude from 30° to 50°, etc.

This construction gives a very close approximation up to 70°.

Fig. 304 is drawn four times the scale of Fig. 303.

Another form of development is the conical. The map is supposed to be projected on to a cone and then straightened out into a plane. This is illustrated in the following problem.

The cone is supposed to encircle or cut the sphere, according to the portion of the sphere developed, and the circle of contact between them is taken as the middle latitude of the map.

Problem 247.

To determine the meridians and parallels of a map by conical projection. Fig. 305.

Draw a line CE to represent the middle meridian of the map. Assume A to be the position of the parallel of 55° and the middle parallel of the map. Let AB = 5° of latitude.

Set off AC = 57·3°, the radius of sphere, and repeat the distance AB above and below A for the positions of the parallels on the middle meridian.

Let us now refer to Fig. 306. This is drawn to half the scale of Fig. 305.

Make AC = 57·3°. With A as centre, and radius AC, draw a quadrant till it meets a perpendicular on A in D. Draw the line AE at an angle of 55° with AC, cutting the arc in F. Draw

a line from D perpendicular to AD till it meets AE in E, also a line from F parallel to ED and cutting AD in G.

Draw CI perpendicular to AC and 10° in length. Join IA. Draw FH perpendicular to AC and cutting IA in J.

Then DE is the co-tangent of the angle 55°, GF is the cosine, and JH = 10° of the latitude at 55°.

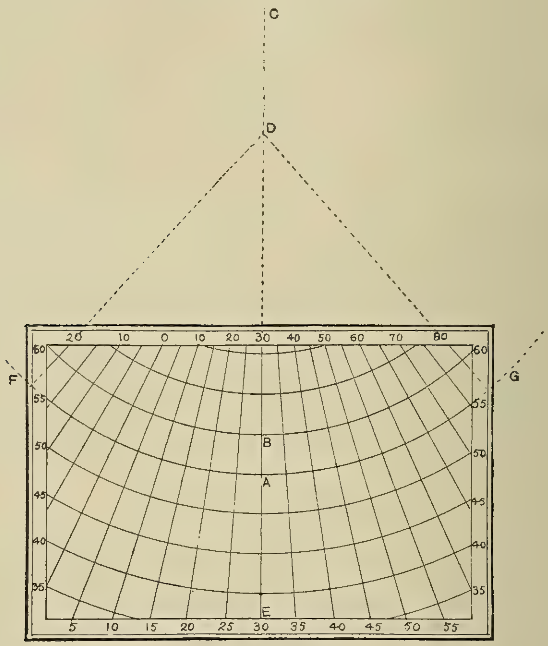

Fig. 305.

Now return to Fig. 305. As the scale of this figure is twice that of Fig. 306, make AD = twice the co-tangent (DE) of Fig. 306. With D as centre, and all the divisions on DE as radii, draw arcs for the parallels of latitude. The arc FG (55°) is the development of the circle of contact between the cone and the sphere.

Set off on this arc, on each side of A, divisions equal to JH (Fig. 306). As Fig. 305 is twice the scale of Fig. 306, each of these divisions will represent 5°.

Draw lines from D through these divisions for the meridians of longitude.

Note.—As the generatrices of a cone are all of equal length, the development of a cone must be the sector of a circle (Prob. 52).

In small maps the meridians and parallels are drawn as straight lines. The meridian passing through the centre of the map is divided in a similar manner to the line DE (Fig. 305); and the extreme parallels, top and bottom, are divided in proportion to their latitude, in the same way as the arc FG (Fig. 305), by the construction shown in Fig. 306.

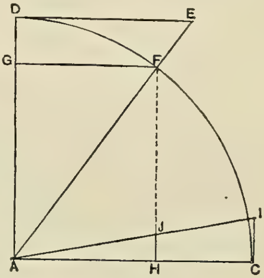

Fig. 306.

Flamstead's projection consists in making all the parallels straight lines instead of arcs, as shown in Fig. 305. The middle meridian is a straight line perpendicular to the parallels; and the proportional divisions for the degrees of latitude are set off from this meridian on each of the parallels, and curves drawn through them for the meridians.

Note.—The proportional divisions for the degrees of latitude could be determined by Fig. 306.

An improvement on this projection is to make all the parallels concentric arcs by conical development (Fig. 305), and then set off on each arc divisions in proportion to the degrees of latitude (Fig. 306), and draw curves through the divisions for the meridians.

CHAPTER XXXVIII

GRAPHIC ARITHMETIC

The geometrical principle of simple proportion is illustrated in Chap. VII. The same principle can also be applied to the other branches of arithmetic, viz. addition, subtraction, multiplication, division, involution, and evolution.

Any magnitude can be considered as a unit. The *unit* is the element in measuring and expresses the standard with which all magnitudes of the same character can be compared.

In the following terms—*ten* inches, *nine* acres, *five* grammes, etc.—the quantity of each article denotes the number of units it contains; inches, acres, grammes, etc., are the units employed; any other quantity of the same kind would equally well serve for the unit, as convenient.

$$\begin{array}{llll} & \text{A line } x \text{ inches in length} & = \text{M} & \text{units.} \\ & \quad ,, \quad \text{one inch} \quad ,, & = \dfrac{\text{M}}{x} & ,, \\ \therefore & \quad ,, \quad y \text{ inches} \quad ,, & = \dfrac{\text{M}y}{x} & ,, \end{array}$$

If N = the number of inches contained in a line x,

$$\begin{array}{lll} \text{A line } x & \text{inches in length} & = \text{N units} \\ \quad ,, \quad \dfrac{x}{\text{N}} & \quad ,, \qquad ,, & = \text{one unit} \\ \quad ,, \quad \dfrac{\text{M}x}{\text{N}} & \quad ,, \qquad ,, & = \text{M units.} \end{array}$$

If a given line be taken to represent a number, *e.g.* a line 4 inches long = 40 units, other numbers could be represented by means of the same scale.

A line two inches long = 20 units
„ three „ „ = 30 „
$\frac{1}{10}$ of the given line = 4 „
$\frac{1}{40}$ „ „ = 1 unit.

ADDITION.

Problem 248.

Find the sum of $5 + 2 + 6$, *the unit being* $\frac{1}{4}''$. Fig. 307.

Draw a line and set off from A to B 5 units, from B to C 2 units, and from C to D 6 units.

Fig. 307.

Then AD is the line required.

SUBTRACTION.

Problem 249.

Find the difference between 9 *and* 4, *or* $9 - 4$; *the unit being* $\frac{1}{4}''$. Fig. 308.

Draw a line AB 9 units in length. From B towards A set off 4 units.

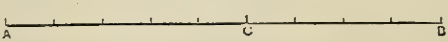

Fig. 308.

Then AC is the line required.

REPRESENTATION OF NUMBERS BY LINES.

Problem 250.

A given line is 9 units in length; determine the unit. Fig. 309.

Let AB be the given line.

Draw AC at any angle with AB, and set off along it from A

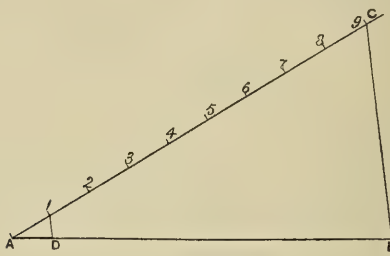

Fig. 309.

9 equal divisions (any convenient size). Join 9 to B. Draw a line from 1 parallel to 9B, meeting AB in D. Then AD is the unit.

Problem 251.

Find a line to represent 5, *when* $\frac{3}{4}'' = 2$. Fig. 310.

Draw a line AB $\frac{3}{4}''$ long, and bisect it in C.

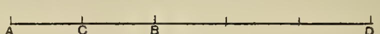

Fig. 310.

Produce AB, and set off from B a distance equal to AC, three times, to determine D. Then AD is the line required.

Problem 252.

A given line AB, $1\frac{1}{2}''$ *long, represents* 11; *produce the line till it represents* 14. Fig. 311.

Let AB be the given line. Draw AC at any angle, and set

off along it from A 14 equal divisions (any size). Join 11B, and

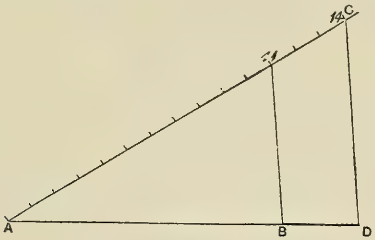

Fig. 311.

draw a line parallel to it from 14 till it meets AB produced in D. Then AD is the line required.

Problem 253.

Determine a line to represent $\frac{4}{5}$, *the unit being* 1·75″. Fig. 312.

Draw AB 1·75″ in length. Draw AC at any angle, and set

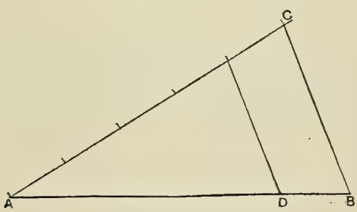

Fig. 312.

off from A 5 equal divisions (any size). Join CB. From 4 draw a line parallel to CB till it meets AB in D. Then AD is the line required.

Problem 254.

Show graphically $5 + 3 - 2 + 4$, *the unit being* $\frac{1}{6}$″. Fig. 313.

Draw a line from A indefinite in length.

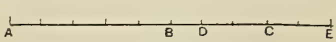

Fig. 313.

Set off AB = 5 units. Set off from B, BC = 3 units. From

C, towards A, set off CD = 2 units. From D, towards C, set off DE = 4 units. Then AE is the line required.

PROBLEM 255.

If a given line A *represents the unit, determine the length of* B. Fig. 314.

Draw a line CE equal to the line B. From C, on the same line, set off CD equal to A. Draw CG at any angle with CE,

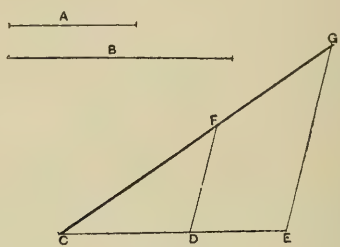

Fig. 314.

and on it set off CF = 1 inch. Join DF, and from E draw a line EG parallel to it. Then CG is the length of B in inches, in this case $1\frac{3}{4}$ ins. Therefore if A represents 1 unit, B will represent $1\frac{3}{4}$ units.

MULTIPLICATION.

When one number is multiplied by another, 6×3 for instance, the answer (18) is the same as 18×1; therefore $6 : 18 :: 1 : 3$, or, as the unit $: 3 :: 6 : x$; consequently if two lengths and the unit are known, we have three terms given, from which we can determine the answer x.

PROBLEM 256.

To determine a line to represent A × B, C *being the unit.* Fig. 315.

Draw a line *de* equal to the unit C. At *e* draw a perpendicular *ef* equal to B. Draw a line from *d* through *f*.

Set off from d, on de produced, dg equal to A. At A draw a perpendicular till it meets df produced in h. Then $gh = \mathrm{A} \times \mathrm{B}$.

$$gh : dg :: fe : de;$$
$$\text{or, } gh : \mathrm{A} :: \mathrm{B} : \mathrm{C}.$$
$$\therefore gh = \mathrm{A} \times \mathrm{B}.$$

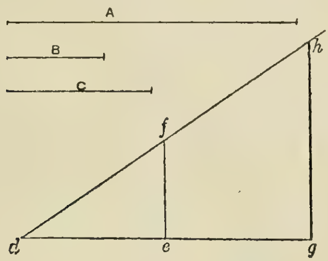

Fig. 315.

Problem 257.

Determine graphically $1\frac{1}{2}'' \times \frac{1}{2}''$, *the unit being* $\frac{3}{4}''$.

The given lines in the preceding problem (256) are drawn this length, so the solution is precisely the same as shown in Fig. 315.

Problem 258.

To determine the product of $\mathrm{A} \times \mathrm{B} \times \mathrm{C}$, D *being the unit.* Fig. 316.

First determine the product of $\mathrm{A} \times \mathrm{B}$, as follows :—
Draw two indefinite lines from d, at any angle.

$de = \mathrm{D}$ (the unit).
$df = \mathrm{A}$. Join fe.
$dh = \mathrm{B}$. Draw gh parallel to fe.
Then $de : df :: dh : dg$;
or, $\mathrm{D} : \mathrm{A} :: \mathrm{B} : x$.
$\therefore dg = \mathrm{A} \times \mathrm{B}$.

Set off $dj = C$. Join eg. From j draw ji parallel to eg.

$$\text{Then } de : dg :: dj : di;$$
$$\text{or, } D : A \times B :: C : x.$$
$$\therefore di = A \times B \times C.$$

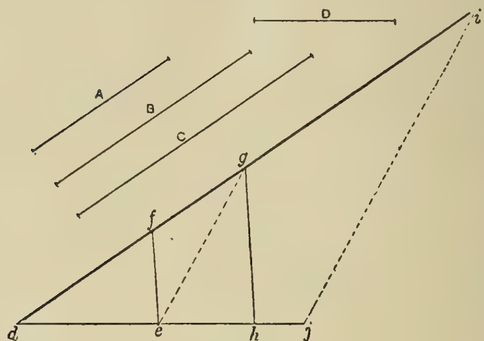

Fig. 316.

PROBLEM 259.

Multiply $\frac{7}{8}'' \times 1\frac{1}{4}'' \times 1\frac{1}{2}''$. *The unit* $= \frac{3}{4}''$.

The given lines in the preceding problem are drawn to these data, so the construction shown in Fig. 316 gives the solution required.

PROBLEM 260.

To determine the continued product of 2, 3, 4, *etc.*
The unit $= \frac{1}{4}''$.

This problem is also constructed in the same way as Prob. 258, but the given lines will have the following lengths: $A = \frac{1}{2}''$, $B = \frac{3}{4}''$, $C = 1''$, and $D = \frac{1}{4}''$

DIVISION.

PROBLEM 261.

To divide a line A *by a line* B. C = *unit*. Fig. 317.

Draw two lines de and df perpendicular to each other.

$de = \text{A}$, $df = \text{B}$, $dg = \text{C}$.

Join fe. From g draw gh parallel to fe.

$\therefore dh : de :: dg : df$.

Then dh is the quotient.

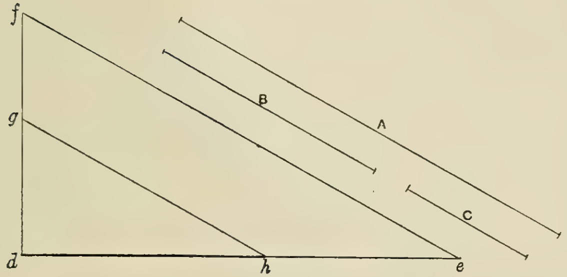

Fig. 317.

Problem 262.

To divide $2\frac{1}{4}''$ *by* $1\frac{1}{4}''$. *The unit* $= \frac{1}{16}''$.

The given lines in the preceding problem are drawn to these data, so Fig. 317 is the solution required.

INVOLUTION.

Involution is the term applied when we multiply a number one or more times by itself. The result is called a *power*.

A number before it is multiplied is called the *first power*, when it is multiplied by itself the *second power*, etc.: *e.g.* first power $= 3$; second power or *square*, 3×3, $= 9$; third power or *cube*, $3 \times 3 \times 3$, $= 27$; etc. These powers are expressed by a small figure above the number; as $3^2 = 9$, or $3^3 = 27$.

If we consider these figures we shall see that involution is a form of simple proportion, *e.g.* $3 \times 3 = 9$, *i.e.* $1 : 3 :: 3 : 9$; so the second power is the third proportional to a given number.

If we continue the proportion—as 3, 9, 27, 81, 243, etc.—we shall find that any number in this series is a third proportional to the two numbers preceding it.

Problem 263.

Determine the square, cube, fourth, fifth, and sixth power of a given line A. B = *unit.* Fig. 318.

Draw from c the lines cd and ce at any angle.

$cf = \text{A}$; $cg = \text{B}$. Join fg.

With c as centre, and radius cf, draw an arc till it meets cd in h. Draw hi parallel to gf.

With c as centre, and radius ci, draw an arc till it meets cd in j. Draw jk parallel to hi.

Then ci = the square, and ck = the cube of A.

Proceed in the same manner to obtain the other powers required.

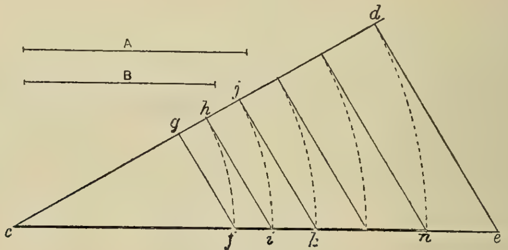

Fig. 318.

Problem 264.

Determine the value of $1\frac{1}{6}''^5$. *The unit* = $1''$.

The given lines in the preceding problem (263) are in accordance with these data, so cn (Fig. 318) is the value required.

Note.—The two preceding problems are based on the *greater third proportional ;* the two following problems are based on the *less third proportional* (Probs. 114 and 115, Plane Geometry).

Problem 265.

Determine the square, cube, fourth, fifth, and sixth power of a fraction ($\frac{7}{9}$). *The line* AB = *the unit.* Fig. 319.

Draw the lines AB and AC at any angle with each other. Make AC = $\frac{7}{9}$ of AB (Prob. 253).

With A as centre, and radius AC, draw an arc till it meets AB in D. Draw DE parallel to BC.

With A as centre, and radius AE, draw an arc till it meets AB in F. Draw FG parallel to DE.

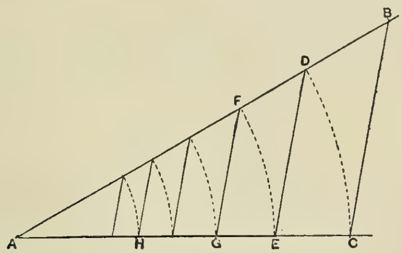

Fig. 319.

Then AE is the square, and AG the cube. The other powers are determined in the same manner.

Problem 266.

Find the value of $\frac{7}{9}^5$. *The unit* = $2\frac{1}{4}''$.

In the preceding problem the line AB = $2\frac{1}{4}''$ and the line AC = $\frac{7}{9}$ of AB, so AH is the value required.

EVOLUTION, OR THE EXTRACTION OF ROOTS.

If a number is multiplied one or more times by itself, the number so multiplied is called a *root*.

The sign $\sqrt{\ }$ (called the radical sign), when placed before a number indicates that its square root is to be extracted, as $\sqrt{9} = 3$. The sign $\sqrt[3]{\ }$ shows that the cube root is to be extracted, as $\sqrt[3]{729} = 9$.

The square root of $9 = 3$, *i.e.* $3 \times 3 = 9$, or stated proportionally it would be $1 : 3 :: 3 : 9$; so it is the mean proportional between the unit and a given number.

Problem 267.

A given line AB *represents* 5; *find its square root.* Fig. 320.

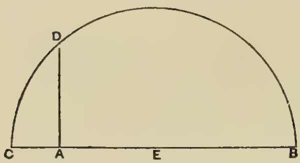

Fig. 320.

Produce AB to C, and make AC $= \frac{1}{5}$ of AB.

Bisect CB in E. With E as centre, and radius EB, draw a semicircle. At A draw AD perpendicular to CB. Then $AD = \sqrt{5}$.

Problem 268.

A given line A *represents the* $\sqrt{7}$; *determine the unit.* Fig. 321.

Assume any unit DB, and produce the line to C, making BC seven times the length of DB.

Bisect DC in E. With E as centre, and radius EC, draw a semicircle.

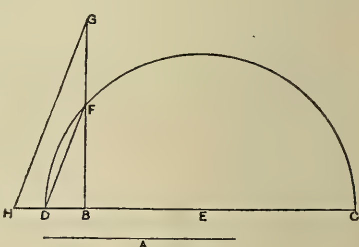

Fig. 321.

Draw BF perpendicular to DC, and join FD. Set off BG on BF produced equal to the given line A. From G draw a line parallel to FD till it meets CD produced in H. Then HB is the unit required.

Problem 269.

To determine the square roots of 2, 3, 4, 5, *etc.*
Unit $= 1''$. Fig. 322.

Draw AB and BC, each 1 inch in length, perpendicular to each other. Join AC.

Then by Euc. I. 47, $AC^2 = AB^2 + BC^2 = 1 + 1 = 2$.

$$\therefore AC = \sqrt{2}.$$

With A as centre, and radius AC, draw an arc till it meets

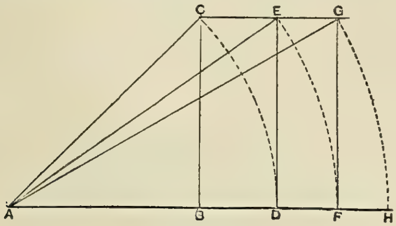

Fig. 322.

AB produced in D. Draw DE, one unit in length, perpendicular to AD. Join AE.

Then $AE = \sqrt{3}$.

Proceed in the same manner to find the square roots of 4, 5, etc.

Problem 270.

To determine $\sqrt{\dfrac{m''}{n}}$ *by construction.* Fig. 323.

Let $m = 7$ and $n = 9$.

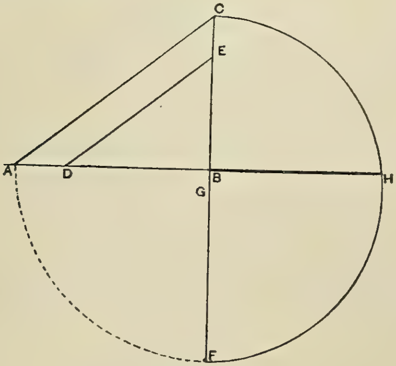

Fig. 323.

Draw AB and BC perpendicular to each other.

Set off BD on $BA = 9$, and BE on $BC = 7$. Join DE.

Set off BA = 1″, and draw AC parallel to DE.

Then BC = $\frac{7}{9}''$. To find its square root :—

Produce CB till BF = 1″. Bisect CF in G. With G as centre, and radius GF, draw a semicircle. Draw BH perpendicular to CF. Then BH = $\sqrt{\frac{m''}{n}}$.

VULGAR FRACTIONS.

Problem 271.

To convert a series of fractions, e.g. $\frac{3}{4}$ *and* $\frac{5}{6}$, *to a common denominator. Unit* = $\frac{1}{6}''$. Fig. 324.

Draw the lines Ac and AF at any angle.

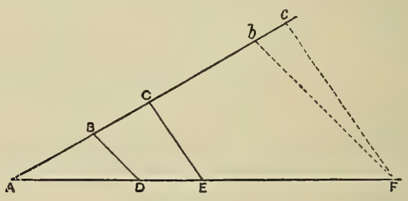

Fig. 324.

Set off AB = 3, and AC = 5, for the numerators; also AD = 4, and AE = 6, for the denominators.

Set off AF = 12 for the common denominator.

From F draw Fb parallel to DB, and Fc parallel to EC. Then $\frac{Ab}{AF} = \frac{3}{4}$, and $\frac{Ac}{AF} = \frac{5}{6}$.

The triangle ABD is similar to the triangle AbF.

$$\therefore \frac{AB}{AD} = \frac{Ab}{AF}.$$

The same thing applies to the triangles ACE and AcF.

If AF represented the unit, then Ab = $\frac{3}{4}$ and Ac = $\frac{5}{6}$ of AF.

Note.—The *addition* and *subtraction* of vulgar fractions could also be performed in the same manner :

$$e.g., \text{ in addition the sum} = \frac{Ab + Ac}{AF},$$

$$\text{in subtraction the difference} = \frac{Ab - Ac}{AF}.$$

PROBLEM 272.

Find the value of $\dfrac{3 \times 5 \times 2}{7 \times 9 \times 5}$.

Multiply $3 \times 5 \times 2 = 30$ (Prob. 258),
also $7 \times 9 \times 5 = 315$.
Then divide 315 by 30 (Prob. 261).

REPRESENTATION OF AREAS BY MEANS OF LINES.

The line *gh* (Prob. 256) shows the number of *linear units* one line multiplied by another contains, *e.g.* A × B.

If A and B represented two adjacent sides of a rectangle, then *gh* would show the number of *units of area*, or square units, the rectangle contains.

The number of units of area contained by any surface can be expressed by a right line whose length is proportional to the extent of the area to be determined.

The areas of the plane figures shown in Fig. 325 are determined as follows:—

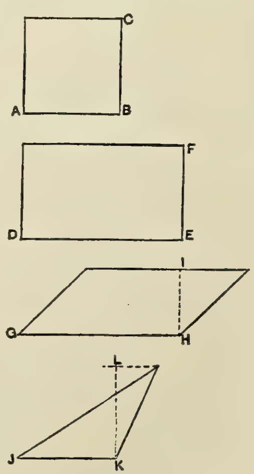

Fig. 325.

Square.—AB × BC.

Rectangle.—DE × EF.

Parallelogram.—One side multiplied by the perpendicular height, *i.e.* GH × HI.

Triangle.—Half of base multiplied by the perpendicular height, *i.e.* half of JK × KL.

To multiply one line by another, see Prob. 256.

In Chap. VIII. problems are shown for converting all the plane figures into triangles. The area of each triangle can then be determined by multiplying half the base by perpendicular height (Prob. 256), or the areas of all plane figures could be cut up into triangles ; the area of each triangle could be determined by Prob. 256, and then adding the products of the triangles together (Prob. 248).

Problem 273.

To determine a line to represent the area of a circle.

Note.—We must first find a right line equal in length to a semicircle (Prob. 192, Plane Geometry). The proportion between the radius and the semicircle is nearly 3·14159 ; this is expressed by the Greek letter π (pi). It is usual in practice to take this as $3\frac{1}{7}$ or $\frac{22}{7}$.

Area of circle $= \pi r^2$, *i.e.* the semicircle multiplied by the square of its radius.

Lines representing the number of units of area contained by the surfaces of the following figures can also be determined by multiplying one quantity by another : if two quantities, by Prob. 256 ; or three quantities, by Prob. 258.

Ellipse.—Major axis × minor axis × 0·7854.

Parabola.—Base × height × $\frac{2}{3}$.

Surface of Cylinder.—Length × perimeter.

Surface of Cone.—Half the perimeter of base × length of generatrix.

Surface of Sphere.—Square of diameter × 3·14159.

REPRESENTATION OF VOLUMES BY MEANS OF LINES.

The line AB (Fig. 325) represents the number of *linear units*; AB^2 would represent the number of *units of area* in the square, and AB^3 the number of *units of volume* in the cube.

Problem 274.

To determine a line to represent the number of units of volume contained by a rectangular solid $5'' \times 4'' \times 3''$. *The unit* $= 1''$.

$5 \times 4 \times 3 =$ number of units of volume (Prob. 258).

Lines representing the number of units of volume contained by the following solids can be determined by Probs. 256 and 258.

Cube is the product of three factors, each of which is the length of one edge: *e.g.* $a =$ edge, $a \times a \times a$, or a^3.

Rectangular solid = product of three dimensions, or area of base $\times$ perpendicular height.

Pyramid = height $\times \frac{1}{3}$ the area of base.

Cone = height $\times \frac{1}{3}$ the area of base.

Cylinder = area of base $\times$ perpendicular height.

Sphere = cube of diameter $\times$ 0·5236, or $\frac{4}{3}\pi r^3$.

To determine the volume of each of the *regular solids*, assume each face to be the base of a pyramid, the centre of the solid being the vertex of each pyramid. Determine the length of each axis by Prob. 222.

Then the axis $\times \frac{1}{3}$ of area of base $\times$ number of faces = volume of solid.

A right line to represent the volume can be determined by Prob. 258.

CHAPTER XXXIX

GRAPHIC STATICS

THE treatment of forces which are so applied to a body that they do not tend to alter its state of rest is called *Statics.*

Force is that which tends to alter the state of rest or motion of a body.

A force may be adequately represented by a straight line, because it has *magnitude* and *direction.* The magnitude of the straight line must represent in linear units the number of units of force, and the direction of the line must represent the direction in which the force is applied.

Two forces can be represented by two straight lines, if the same fixed standards, or units of measurement, are used for both. The unit of weight or pressure may be either a gramme, a grain, a pound, etc.; and these units could be expressed by any linear measurement, as a fraction of an inch, or of a foot, etc.

PROBLEM 275.

Represent by lines, two forces, P *and* Q, *acting at a point* A *at an angle of* 60°. $P = 7$ *lbs, and* $Q = 10$ *lbs. Unit* $\frac{1}{8}'' = 1$ *lb.* Fig. 326.

Draw PA, $\frac{7}{8}''$ long, and QA at an angle of 60° with it, $1\frac{1}{4}''$ long.

Parallelogram of Forces.—If two forces acting at a point be represented in magnitude and direction by two straight lines, and these lines be taken to represent two adjacent sides of a parallelogram, the diagonal of the completed figure through the point represents, in magnitude and direction, a force which may be substituted for them, called their *resultant.*

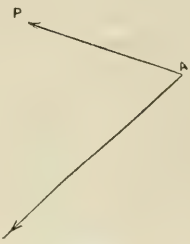

Fig. 326.

PROBLEM 276.

Draw a line to represent the resultant of the forces P *and* Q *of the preceding problem.* Fig. 327.

Draw PB parallel and equal to AQ. Join BQ and AB. Then the line AB, measured by the same unit ($\frac{1}{8}''$), represents

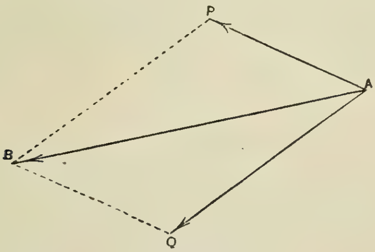

Fig. 327.

the resultant; *i.e.* a force equal to AB, applied in the direction B, is equivalent to the combined forces P and Q acting in the directions of P and Q.

Composition of Forces.—Two forces can be combined into a single force,—the *resultant,*—the separate forces being called the *components.*

Resolution of Forces.—Any original force can be assumed to be the diagonal of a parallelogram, and can be resolved into two forces acting in any desired direction.

Problem 277.

Let AB *represent a force of* 12 *lbs. acting in the direction* AB; *it is required to resolve it into two component forces acting parallel to the given lines* C *and* D. *Unit* $\frac{1}{8}'' = 1$ *lb.* Fig. 328.

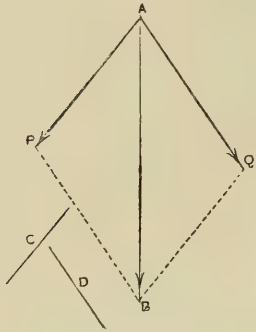

Fig. 328.

Draw AB $1\frac{1}{2}''$ long. Draw the lines AP and BQ parallel to C, and lines from A and B parallel to D, till they meet those drawn parallel to C in P and Q. Then AP and AQ are the forces required.

Problem 278.

A force of 14 *lbs. has two components acting at an angle of* 30° *with each other; one of these forces is* 8 *lbs., determine the other. Unit* $\frac{1}{8}'' = 1$ *lb.* Fig. 329.

Draw AP, 8 units in length, and AQ at an angle of 30° with it. Draw PB parallel to AQ. With A as centre, and radius of 14 units, draw an arc cutting PB in B. Draw BQ parallel to PA, and cutting AQ in Q. AQ is the force required.

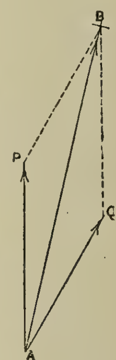

Fig. 329.

Problem 279.

A force of 9 *lbs. has two components, in the proportion of* 4 : 3, *perpendicular to each other; determine their values. Unit* $\frac{1}{8}''$ $= 1$ *lb.* Fig. 330.

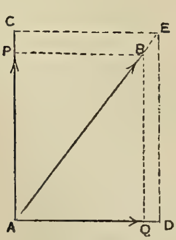

Fig. 330.

Draw the lines AC and AD perpendicular to each other in the given proportion (4 : 3). Draw the diagonal AE. From A set off AB

= 9 units, and draw BP and BQ parallel to AD and AC. Then AP and AQ are the values required.

Problem 280.

A vessel is sailing due South; the wind, equal to force P, *is blowing from the South-West; show the proportion of this force that will act parallel to the keel when the sail is set in the position* AB. Fig. 331.

Resolve the given force P into two components —one (r) acting perpendicularly to the sail, the other (p) acting in the plane of the sail: this latter component is lost. Produce r, and set off r' equal to it. Resolve r' into two components, one of which (R) acts parallel to the keel and is the force required: the other component (q) has little effect on account of the lateral resistance of the vessel.

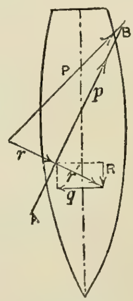

Fig. 331.

Problem 281.

A weight, A, *of* 24 *lbs. is suspended by a cord,* 10 *feet in length, at point* B, *and is pulled* 6 *feet from its vertical position by a horizontal force; determine the tension on the cord. Unit* $\frac{1}{8}'' = 1$ *foot.* Fig. 332.

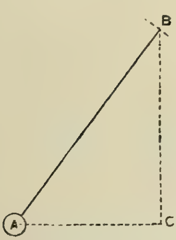

Fig. 332.

Draw AC 6 feet long. With A as centre, and radius of 10 feet, draw an arc till it meets a perpendicular on C in B.

Let BC, which measures 8 feet, represent the vertical force of 24 lbs.

If 8 feet = 24 lbs., 1 foot = 3 lbs., and 10 feet = 30 lbs.,

∴ the tension required is 30 lbs.,

the force BA being the resultant of the forces BC and CA.

Fig. 333 is a practical illustration of the resolution of forces. P = 1 cwt. acts in the direction of AB; it is required to know what weight is sustained by each of the supports C and D.

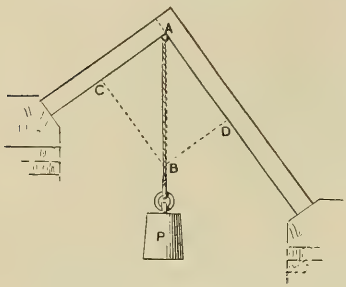

Fig. 333.

At any point B on AB draw lines BD and BC parallel to AC and AD. If AB be taken to represent 112 units, then the number of units to the same scale in BC gives the number of lbs. supported by D, and the number of units in BD gives the number of lbs. supported by C.

Problem 282.

Three forces, P, Q, *and* R, *act at a point* A, *at right angles to each other;* P = 4 *lbs.,* Q = 6 *lbs., and* R = 9 *lbs.; determine their resultant.* *Unit* $\frac{1}{8}'' = 1$ *lb.* Fig. 334.

Draw QB parallel to AR till it meets a perpendicular on R in B. Join AB. AB is the resultant of the forces Q and R. Draw PC parallel to AB, and join AC, which is the resultant required.

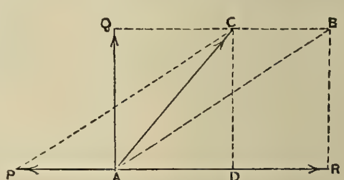

Fig. 334.

Note.—Two equal forces acting at a point, in opposite directions in the same straight line, are in equilibrium. The resultant of two unequal forces, acting in opposite directions, in the same straight line, is the difference between the forces; *e.g.* the difference between the forces P and R (Fig. 334) is AD, and AC is the resultant of the forces AD and Q.

PROBLEM 283.

Find graphically the resultant of three forces, P, Q, *and* R, *acting at a point* A. *Unit* $\frac{1}{8}'' = 1$ *lb.* Fig. 335.

Draw RB parallel to AQ till it meets a line from Q, parallel to AR, in B. Join AB. Draw a line from P, parallel to AB, till it meets a line from B, parallel to AP, in D. Join AD, which is the resultant required.

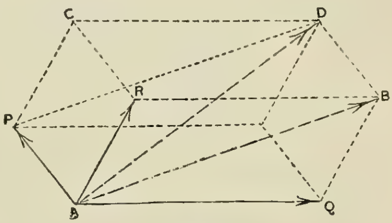

Fig. 335.

Draw a line from P, parallel to AR, till it meets a line from R, parallel to AP, in C. Join CD.

The main lines of the figure represent a *parallelopiped,* and the principle is called the *parallelopiped of forces.*

The Triangle of Forces.—If three forces acting at a point are represented in magnitude and direction by the sides of a triangle *taken in order,* the three forces are in equilibrium. Any one of the forces is equal in magnitude, and in the opposite direction, to the resultant of the other two.

PROBLEM 284.

Let a given triangle ABC *represent three forces in equilibrium; show how these forces act at a point* A. Fig. 336.

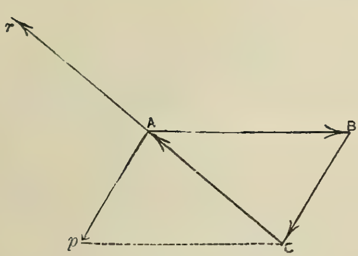

Fig. 336.

Draw Ap parallel and equal in length to BC. Produce CA to r, making Ar equal in length to AC. Then AB, Ap, and Ar

are the three forces in equilibrium, acting at the point A. If we reverse the arrow-head on AC, it will be the resultant of the forces AB and Ap, being directly equal and opposite to the third side of the triangle CA, as AC.

Problem 285.

Three given forces, p, B, and r, are in equilibrium; construct a triangle of forces from them. Fig. 336.

This is the converse of the preceding problem.

Draw each side of the triangle parallel and equal to each of the forces.

The Polygon of Forces.—If any number of forces acting at a point are represented in magnitude and direction by the sides of a polygon *taken in order*, then these forces are in equilibrium. Any one of the forces is equal in magnitude, and in the opposite direction, to the resultant of the other forces.

Problem 286.

Let p, q, r, s, and t represent a series of forces in equilibrium, acting at point A; *construct a polygon of forces from them.* Fig. 337.

Find the resultant of the forces p and t, as AB. Find the resultant of the forces AB and s, as AC; also the resultant of AC and r, as AD.

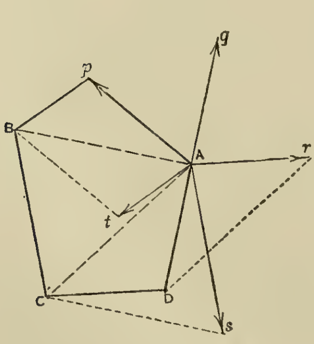

Fig. 337.

Then pBCDA is the polygon required. AD is directly opposite and equal to the remaining force q. The same thing would apply if any of the other forces were taken as the remaining force.

The student should notice that each side of the polygon is

parallel and equal to one of the forces, *e.g.* BC is parallel and equal to s, etc.; and that when the forces are in equilibrium, their directions follow one another round the polygon.

Problem 287.

The following forces, given in direction, viz. $p = 7$ *lbs.*, $r = 6$ *lbs.*, $s = 9$ *lbs.*, *and* $t = 5$ *lbs.*, *act at a point* A; *give the geometrical construction for finding the resultant. Unit* $\frac{1}{10}''$ $= 1$ *lb.*

Using the same Fig. (337), draw the following lines:—pB equal and parallel to force t, BC equal and parallel to force s, and CD equal and parallel to force r. Then the line DA, which completes the polygon, represents in magnitude and direction the force which is in equilibrium with the four given forces. The resultant of the forces is therefore equal and opposite to DA.

Note.—When a series of forces in the same plane act at a point, and the polygon of the forces closes, they must be in equilibrium; and if it does not close, the side which is wanting is equal and opposite to the resultant, and will restore equilibrium.

Moments of Force.—The moment of a force about a point is the product of the force and the perpendicular drawn to its direction from the point. A moment represents the tendency of a force to turn a rigid body on an axis through the point.

Parallel Forces.—When a rigid body is in equilibrium under parallel forces, the sum of those in one direction is equal to the sum of those in the other.

When two parallel forces act in the same direction, the resultant is equal to their sum, and is called the *resultant of like forces;* when they act in the opposite direction, their resultant is equal to their difference, and is called the *resultant of unlike forces.*

Problem 288.

To express graphically the moment of a force P *with respect to a point* C. Fig. 338.

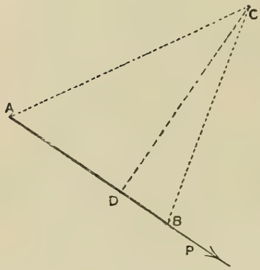

Fig. 338.

Draw a line from C perpendicular to the direction of the force AP, and meeting it in D. Set off AB to represent the magnitude of the force. Then AB $\times$ DC = the moment of the force P. This moment can be expressed by a line (Prob. 256).

The moment of the force P is equal to twice the area of the triangle ABC.

Problem 289.

Two parallel forces, $P = 6$ *lbs., and* $Q = 4$ *lbs., act at the extremities of a rigid bar* AB; *determine the resultant, and the point of application on* AB *to secure equilibrium. Unit* $\frac{1}{10}'' = 1$ *lb.* Fig. 339.

The line AB must be first divided inversely as the forces. As $P + Q = 10$; divide AB into 10 equal parts, 6 of which will give the distance of the required point of application (C) from B.

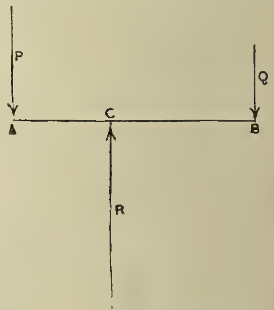

Fig. 339.

As P and Q are *like forces*, the resultant $R = P + Q = 10$.

The moments of the forces P and Q about C are equal, *i.e.* $P \times AC = Q \times CB$.

Note.—In the two following problems it is advisable to use two scales—one of *linear units*, and the other *units of force*.

Problem 290.

Two forces, P = 9 *lbs., and* Q = 6 *lbs., act in opposite directions on a rigid bar* AB, *in the points* C *and* B, 4 *linear units apart; determine the resultant* R *and its point of application to secure equilibrium. Unit of force* $\frac{1}{10}''$ = 1 *lb. Linear unit* $\frac{1}{8}''$. Fig. 340.

These being *unlike forces,* the resultant R = P − Q = 3 lbs.

The forces R and Q must be applied on opposite sides of the larger given force P, and their moments about C must be equal;

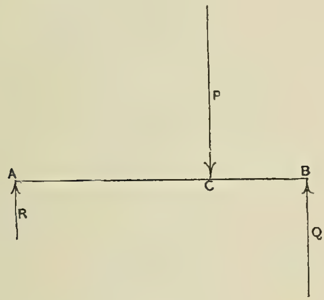

Fig. 340.

that is, R × AC, which tends to turn the bar in the same direction as the hands of a watch, must be equal to Q × BC, the moment in the opposite direction.

$$AC = \frac{Q \times BC}{R} = \frac{6 \times 4}{3} = 8 \text{ linear units.}$$

Problem 291.

A given force, P, 9 *lbs., acts on a rigid bar* AB *at the point* C; *resolve it into two parallel forces* R *and* Q, *their points of application to be* A *and* B. *Linear unit* $\frac{1}{8}''$. *Unit of force* $\frac{1}{10}''$ = 1 *lb.* Fig. 340.

This problem is illustrated in Fig. 340. The relative proportion of R and Q is determined as follows :—

$$R = \frac{P \times CB}{AB}, \text{and } Q = \frac{P \times AC}{AB}.$$

Problem 292.

To determine the resultant of three parallel forces, P, Q, *and* S, *acting on a rigid body in the points* A, B, *and* D. P = 4 *lbs.*, Q = 2 *lbs.*, *and* S = 3 *lbs. Unit of force* $\frac{1}{8}'' = 1$ *lb.* Fig. 341.

Divide AB inversely as the forces P and Q. As P + Q = 6, divide AB into six equal parts, two of which will give the distance AC. Then P × AC = Q × CB. The resultant r = P + Q = 6. We have now to find the resultant of the forces r and S in the same manner: r + S = 9. Divide CD into nine equal parts, three of which will give the distance CE. Then r × CE = S × DE. The resultant R = P + Q + S acting at the point E. This point is called the *centre of the parallel forces.*

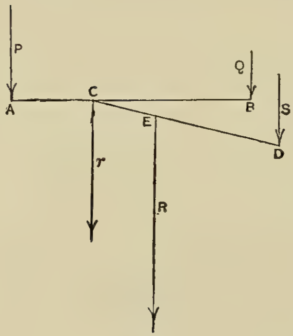

Fig. 341.

Problem 293.

Draw an equilateral triangle ABC *with* $1\frac{1}{8}''$ *sides. Let* P = 6 *lbs.*, Q = 5 *lbs.*, *and* S = 3 *lbs.*, *be parallel forces acting at the points* A, B, *and* C. *Determine the resultant force. Unit* $\frac{1}{8}'' =$ 1 *lb.* Fig. 342.

First determine the resultant of the forces S and Q. As S + Q = 8, divide CB into eight equal parts, and at a distance of three of these divi-

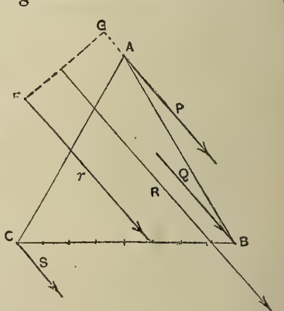

Fig. 342.

sions from B draw the resultant r parallel to the other forces and equal in length to $S + Q$ (8).

Draw a line FG perpendicular to r, and meeting the force P produced. Next determine the resultant of the forces r and P. As $r + P = 14$, divide the line FG into fourteen equal parts, and at a distance of eight of these divisions from G draw R parallel to the other forces and equal in length to $r + P$ (14). This is the resultant required.

Problem 294.

Draw an equilateral triangle ABC *with* $1\frac{1}{8}''$ *sides. Let a force,* $P = 5$ *lbs., act from* A *to* B; *a force,* $Q = 4$ *lbs., act from* C *to* B; *and a force,* $S = 4$ *lbs., act from* A *to* C. *Determine the resultant of the forces. Unit* $\frac{1}{8}'' = 1$ *lb.* Fig. 343.

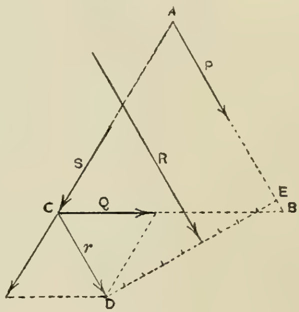

Fig. 343.

First determine the resultant r of the forces S and Q. Draw a line DE perpendicular to the forces r and P. As $r + P = 9$, divide DE into nine equal parts, and at a distance of four of these divisions from E, draw R parallel to the forces r and P, and equal to their sum (9). This is the resultant required.

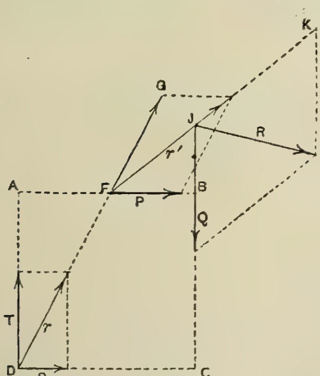

Fig. 344.

Problem 295.

Draw a square ABCD *with* $\frac{7}{8}''$ *sides. Let a force,* $P = 3$ *lbs., act from* A *to* B; $Q = 5$ *lbs., act from* B *to* C; $S = 2$ *lbs., act from* D *to* C; *and a force,* $T = 4$ *lbs., act*

from D *to* A. *Determine the resultant of the forces. Unit* $\frac{1}{8}'' = 1$ *lb.* Fig. 344.

First determine the resultant r of the forces T and S. Produce r, and make FG equal to it. Determine the resultant (r') of the force represented by FG and the force P. Make JK equal to the force r'. Find the resultant (R) of the force represented by JK and the force Q. This is the resultant required.

Problem 296.

Two given forces, P *and* Q, *are applied at a point* A; *show that their moments about any point in their resultant are equal and opposite in direction.* Fig. 345.

Complete the parallelogram of forces ABCD, and take any point a on the resultant AC, and from it draw perpendiculars on P and Q. Then the moment of P about a is represented by ab multiplied by P (Prob. 288), and the moment of Q is in the same way represented by $ad \times$ Q.

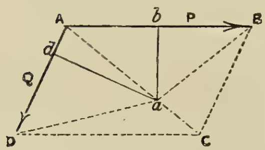

Fig. 345.

Now $ab \times$ P is equal to twice the triangle AaB, and $ad \times$ Q twice the triangle AaD; and since the triangle AaD is equal to the triangle AaB, these moments are equal, and they tend to turn the body in opposite directions.

Moments are called *positive* or *negative* according to the direction of their rotation: if in the direction in which the hands of a clock move, they are called *positive;* and if in the opposite direction, *negative.*

Problem 297.

Draw an equilateral triangle ABC, *with* $1\frac{1}{8}''$ *sides. Let a force,* P = 6 *lbs., act from* A *to* B, *and a force,* Q = 7 *lbs., act from* C *to* A.

1. *Determine the moment of each force with reference to a point* D *in the middle of the line* CB.
2. *Determine a point about which each force has equal moments of opposite signs. Unit* $\frac{1}{8}'' = 1$ *lb.* Fig. 346.

1. Draw DE and DF perpendicular to the forces P and Q. Then P × DF and Q × DE are the moments of the forces.

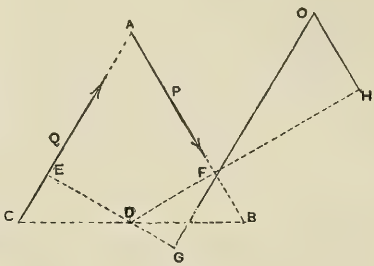

Fig. 346.

2. Set off from E, on ED produced, EG = 6 units. Also set off from F, on DF produced, FH = 7 units. Draw a line from G parallel to Q, and a line from H parallel to P, till they meet in O. This is the point required.

Problem 298.

A rectangular plate ABCD, 10 *feet* × 4 *feet, weighing* 112 *lbs., is suspended from the middle of its upper edge. Assume a weight of* 28 *lbs. fixed to point* A. *Determine the inclination of the plate. Unit* $\frac{1}{8}'' = 1$ *lb.* Fig. 347.

Let E be the point of suspension, and F the centre of the plate. Join AF. We have a force of 112 lbs. acting at F, and a force of 28 lbs. acting at A (28 : 112 : : 1 : 4). Divide AF into five equal parts. From point G, one of these divisions from F, draw a line to E.

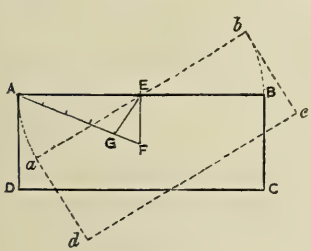

Fig. 347.

When the plate is free to turn this line becomes vertical. Join EF, and set off the line *a*E*b* for the upper line of the plate, making the angle AE*a* equal to the angle GEF. Arcs drawn from the centre E, at distances A and B, will give the positions of *a* and *b*.

Complete the rectangle *abcd*. This will give the new position of the plate.

The *Resultant Moment* of several forces acting at a point in the same plane is the algebraic sum of the moments of the several forces.

Equality of Moments.—The equality of moments is the principle on which all mechanical contrivances are based. This principle regulates the action and constitutes the efficiency of all the *mechanical powers*. It could be illustrated by the *principle of work*—the force multiplied by the distance through which its point of application moves.

Let the bar AB (Fig. 348) be rotated about the centre C, as

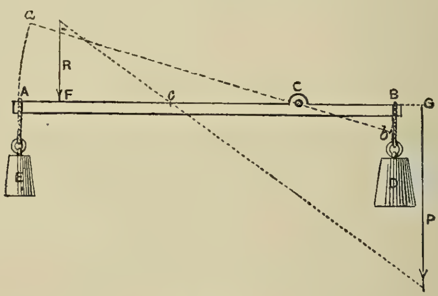

Fig. 348.

shown by the dotted arcs A*a* and B*b*, and let CB equal one-third of AB. Then the arc A*a* will be found to be three times the length of the arc B*b* ; for the length of the arcs are directly as their distances from the centre C.

Let D represent a weight of 12 lbs., and the arc B*b* 2 inches, then $12 \times 2 = 24$. As the arc A*a* is 6 inches in length, the weight of E must be 4 lbs. $6 \times 4 = 24$. The work done by the forces is then equal.

So the lengths of the arcs are inversely proportional to the forces; *e.g.* if B*b* were taken to express the weight of E, A*a* would express the weight of D.

Problem 299.

Let P *represent the magnitude of a force acting on the bar* AB *at the point* G. *Determine the force acting at point* F *which will balance it.* Fig. 348.

Set off F*c* equal to CG. Draw a line from the extremity of P till it meets a perpendicular on AB at F. This determines the length of R, the force required.

Problem 300.

A cord hangs from two fixed points A *and* B, *and supports three weights,* P, Q, *and* R, *fastened to it at the points* C, D, *and* E; *the weight* P = 20 *lbs. Determine the weights of* Q *and* R, *so that the cord shall hang in a given position; also the tension on each division of the cord. Scale* $\frac{1}{10}'' = 5$ *lbs.* Fig. 349.

This is a practical illustration of the triangle of forces; *i.e.* C is a point acted upon by the forces AC, PC, and DC.

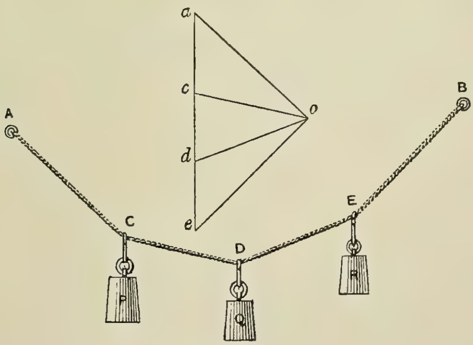

Fig. 349.

Draw *ac* equal to the weight P (20 units), and *ao* and *co* parallel to AC and CD. Then *aoc* is the triangle of forces acting on the point C, and *ao* and *co* measured by the same scale as *ac* will determine the tension on AC and CD.

Produce *ac*, and draw the lines *do* and *eo* parallel to the lines DE and EB. Then *cd* and *de* = the weights Q and R, and *do* and *eo* the tension on the lines DE and EB.

The Funicular Polygon.—When a series of forces in the same plane do not act at the same point, but so as to stretch an endless cord at different points, the polygon formed by the cord then in equilibrium is called the *funicular polygon* for those forces.

PROBLEM 301.

From a series of given forces p, q, r, s, and t, in equilibrium, to determine a funicular polygon. Fig. 350.

Draw a line AB parallel and equal to the given force *p*;

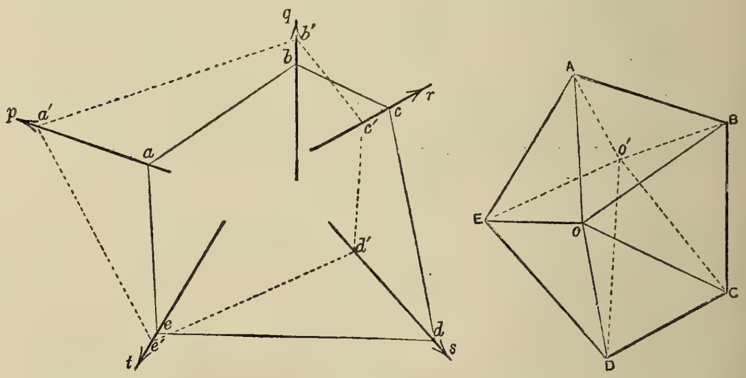

Fig. 350.

BC to *q*; CD to *r*; DE to *s*; and EA to *t*. Assume any point *o* in this polygon,—this is called the *pole*,—and join each of the angles of the polygon to it. These lines are called *vectors*.

Commencing at any point *a* on the line of the force *p*, draw *ab* parallel to the vector *o*B; *bc* to *o*C; *cd* to *o*D; *de* to *o*E. Then if a line be drawn from *e* parallel to the vector *o*A, it will meet the line *ba* on the line of the force *p*, if the forces are in equilibrium.

Note.—Any pole could be chosen, consequently there can be any number of funicular polygons for the same series of forces, if o' were taken for the pole; then $a'b'c'd'e'$ would be the polygon.

The funicular polygon is a device for ensuring that the points of application of the forces should be such that they, when acting together, have no moment about any point, called the *pole.* If the funicular polygon did not close, they would have a moment, which would be balanced by a force represented by the open space. The lines joining the forces are parallel to the vectors forming the sides of the triangles, and it will be seen that the areas of the triangles, whose vertices are the pole, represent the moment of these forces about the pole. This pole may be placed anywhere, for if the forces are in equilibrium they must not tend to turn the body about any point.

The Line of Load, or Load Line.—When the forces are parallel to each other, lines drawn from a point parallel to the forces would result in a single straight line instead of a polygon. To distinguish the forces acting in one direction from those acting in the other, it is usual to use a double line; the downward forces being set off on one line, and the upward forces on the other. This represents a scale of forces, and is called the *load line.*

Problem 302.

A given bar AB, *weighing* 12 *lbs., is hinged at* A, *and is acted upon by a force* $p = 9$ *lbs.; determine where a parallel force,* $r' = 10$ *lbs., must be applied to secure equilibrium. Unit* $\frac{1}{10}'' = 1$ *lb.* Figs. 351 and 352.

Let AB (Fig. 351) represent the bar, and p the given force. Draw a double line *ad* (Fig. 352) to represent the scale of forces. There are four of these to be considered, viz. the weight of the bar, which we will call q; the given force p—both of these are downward forces, so they are set off on one side of the scale;

—the given force r; and the unknown force acting at the hinge which we call s—these two forces act upwards, and are set off on the other side of the scale; their sum must equal the sum of the downward forces.

$$q + p = 21 - r = 11 = \text{the force } s.$$

Assume any point o for the pole, and draw vectors to the

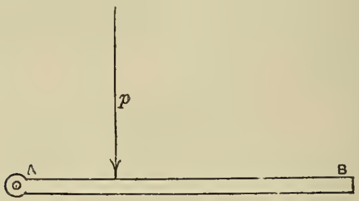

Fig. 351.

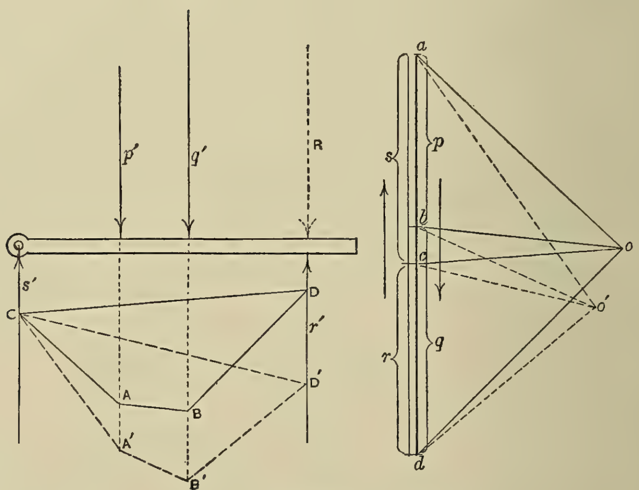

Fig. 352.

points a, b, c, and d. Draw the bar with the force p' in the given position; also q', the weight of the bar, acting through its centre. Draw the force s' acting through the hinge.

At any point C on the line of the force s', draw CA parallel to the vector ao; AB parallel to bo; and BD parallel to do till it meets a line from C parallel to co in D.

Then CABD is the funicular polygon, and D is the point through which the given force r' would act to secure equilibrium.

If o' were taken for the pole, then CA′B′D′ would be the funicular polygon, and it will be seen that the position of the force r' is the same in each instance.

Note.—Any of these forces will be found to be opposite and equal to the resultant of the other three; e.g. $p + q - s = R$, which is equal and opposite to r'.

Couples.—Two equal and opposite parallel forces applied to a body tend to turn it round an axis perpendicular to their plane; this is called a *couple.*

Moment of a Couple.—The amount of the tendency which a couple has to turn a body round an axis, *i.e.* its *torsional value,* is measured by the product of one of the forces and the perpendicular distance between them; *e.g.* P × BC (Fig. 353).

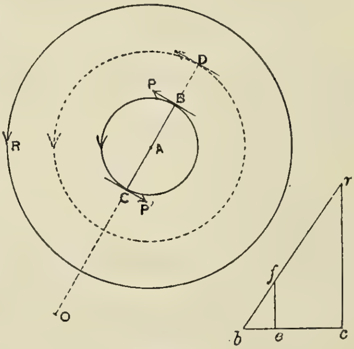

Fig. 353.

Considering P alone, as to its *torsional value* (or tendency to turn a body about A); this is represented by the product of P and the arm AB, at the extremity of which it works. If now P be applied at D (twice its former distance from A), its torsional value about A is equal to that of the sum of the two forces P and P′ acting at the points B and C. Hence it is seen that the torsional value of the couple is equal to the moment of one of the forces about the other end of the arm of the couple, *i.e.* the product of P and BC.

To show graphically the amount of the torsional value of a couple. Let the force P = $1\frac{1}{2}$ lbs. Draw *bc* equal to BC, and set off *be* equal to two-thirds of P, *i.e.* 1 lb. At *e* draw a perpendicular *ef* = P, and draw a line from *b* through *f* till it meets the perpen-

dicular at c in r. Then $cr = \mathrm{BC} \times \mathrm{P}$, $bc =$ the unit (Prob. 256). With A as centre, and radius equal to cr, draw the circle R, the radius of which expresses the torsional value of the couple PP′ in lbs.

The moment of the forces constituting any couple about any point in the plane is equal to the moment of the couple.

Let O be the point. Draw a perpendicular to the forces from it. Then the moment of the couple about O is the difference between $\mathrm{OB} \times \mathrm{P}$, and $\mathrm{OC} \times \mathrm{P}'$; and since P and P′ are equal, the moment about O is $\mathrm{BC} \times \mathrm{P}$, the moment of the couple.

The resultant couple of a series of couples acting in the same plane can be determined in the following manner :—

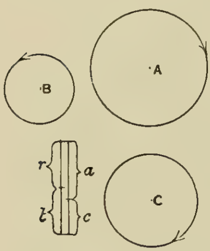

Fig. 354.

Let a couple of the torsional value of 5 units act at A (Fig. 354), one of 3 at B, and of 4 at C. Unit $= \frac{1}{16}''$.

Draw two parallel lines to represent a *line of load*. Set off $a =$ radius of A, and $c =$ radius of C. As these are forces acting in a positive direction, they are set off on the same line. Also set off $b =$ radius of B. Then the remaining length r of the load-line represents the resultant couple required. These two couples are in a negative direction, and balance the two opposite couples ($a + c = b + r$).

Construction of Stress Diagrams.—When rigid bars are connected together at their ends by means of smooth pins, they are said to form a *framework*.

Let AB (Fig. 355) represent a rigid bar held by the pins

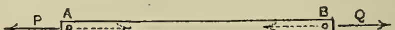

Fig. 355.

A and B, and let P and Q represent the magnitude of the forces at the pins. If the bar is in equilibrium, the forces must be equal and opposite, and must act along the bar. The action of these forces is called a *stress*.

Note.—The weights of the bars are not taken into account.

A convenient method for indicating the forces in stress diagrams is shown in Fig. 356, and is called *Henrici's Notation.*

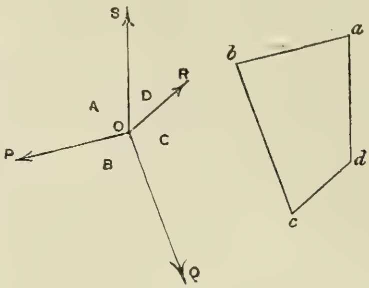

Fig. 356.

Let P, Q, R, and S represent forces acting at the point O, and A, B, C, and D the spaces between the forces. Then the force OP is represented by the letters AB, OQ by BC, etc. Draw *ab* parallel to OP, *bc* to OQ, etc. Then *abcd* is the force polygon representing the forces acting at O.

Note.—The student should notice that the order of the letters is taken in a negative direction, *i.e.* in the opposite direction in which the hands of a clock move. If the order were taken in a positive direction, then BA would represent a force equal and opposite to AB, etc.

Let a triangle PQR (Fig. 357) represent a framework supported at the points P and R, and weighted at Q.

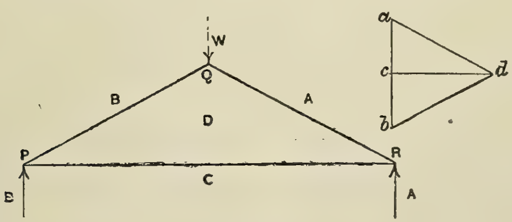

Fig. 357.

Draw a polygon of forces as follows. Draw *ab* to represent the magnitude of the force W.

The forces at P and R are each equal to half of W, so bisect *ab* in *c*. Draw lines from *a* and *b* parallel to the bars QR and PQ, and meeting in *d*. Join *cd*.

The length of each line in the polygon of forces, measured

by the same scale as *ab*, will determine the magnitude of the forces acting upon each of the bars and pins; *e.g.* the force of the bar PQ acting on the pin Q is represented by the line *bd*, and the force of the bar QR on the pin Q is represented by *ad*. These bars are called *struts*, as they are subjected to compression.

The action of the bar PR on the pin P is represented by the line *cd*; *dc* represents the action of P on PR. This bar is subjected to tension, and is called a *tie*.

Problem 303.

To determine the forces acting on the bars and pins forming the framework PQRST. Fig. 358.

Let the framework be supported at the points P and R, and weighted at Q.

Construct the force polygon as follows. Draw *ab* to represent

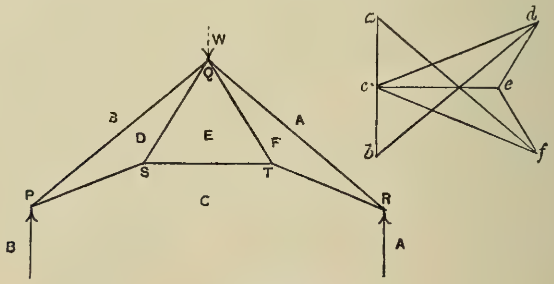

Fig. 358.

the weight W, and bisect it in *c*. Then *ac* and *cb* represent the magnitude of the supporting forces at P and R. Draw lines from *a* and *b* parallel to the bars QR and PQ; also from *c* parallel to the bars PS and TR, meeting the lines from *a* and *b* in *d* and *f*. Draw lines from *d* and *f* parallel to the bars QS and TQ, meeting in *e*. Join *ce*. Then the lengths of the various lines in the polygon of forces, measured by the same scale as *ab*, determine the magnitude of the forces acting upon the corresponding bars and pins.

Note.—The student should notice that the letters in the

polygon of forces correspond to the forces lettered in the framework, as illustrated in Fig. 357.

Problem 304.

To determine the polygon of forces for a series of bars forming a lattice girder. Fig. 359.

This framework consists of two horizontal bars connected by cross bars equally inclined to the vertical.

Let the framework be loaded with equal weights at the

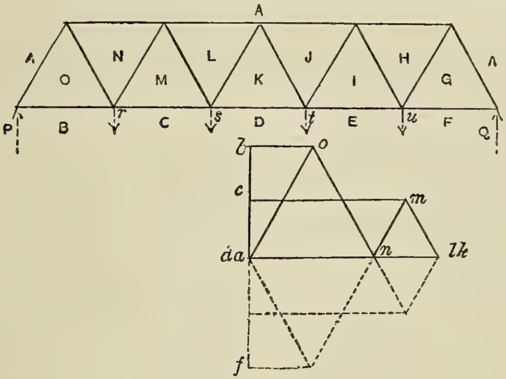

Fig. 359.

points r, s, t, and u, and supported at the points P and Q. Draw bf to represent the sum of the weights, and bisect it in d. Then db and df will represent the supporting forces at P and Q. Draw lines parallel to the several bars, as shown, to determine the forces acting on the various pins, etc. The forces acting on the pin s are LM, MC, CD, DK, and KL. These are shown in the polygon of forces by the lines lm, mc, cd, and dk. To determine the force KL, we draw through l a line lk parallel to kl, and a line through d parallel to DK. These two lines intersect in l, consequently the points l and k coincide, showing that the bar KL has no stress.

The dotted lines in the lower part of the polygon show the forces for the opposite end of the framework.

Problem 305.

To determine the forces acting on the bars and pins forming the framework PQRST. Fig. 360.

Let the framework be supported at the points P and T, and let the weight W be attached to the point R.

First draw a line ab to represent the magnitude of W. Assume any pole O, and join it to the points a and b.

Next draw a funicular polygon as follows. Draw lines from

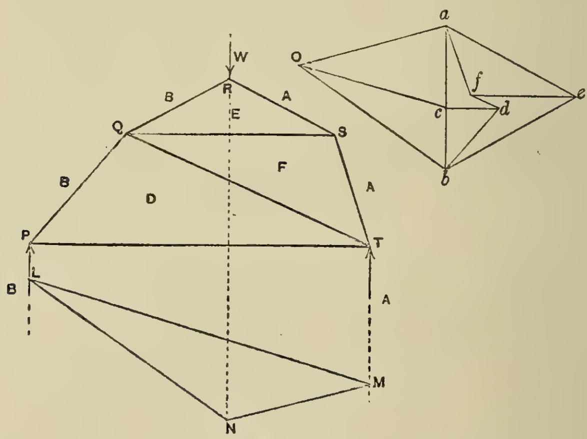

Fig. 360.

P, R, and T, perpendicular to PT. At any point L on the line from P draw LN parallel to Ob. Draw NM parallel to Oa, and join LM. Draw Oc parallel to LM.

The funicular polygon enables us to determine the magnitude of the forces BC and CA acting at the points P and T: *i.e.* the force ab may be resolved into the forces aO acting in the direction MN, and bO acting in the direction NL; the force BC ($=bc$) into forces acting in the directions NL and LM; and the force CA ($=ca$) into forces acting in the directions NM and ML. So the forces BC and CA are represented by the lines bc and ca.

The force polygon is constructed as follows:—Draw lines

from a and b parallel to the bars RS and QR, meeting in e; from a and c, lines parallel to the bars ST and SQ, meeting in f; also from b and c, lines parallel to the bars PQ and PT, meeting in d. Join df. Then the force polygon gives the magnitudes and directions of the forces acting on the several bars and pins.

Problem 306.

To determine the forces acting on the bars and pins forming the framework PQRST. Fig. 361.

In this example two weights, W and W′, are attached to the points Q and R.

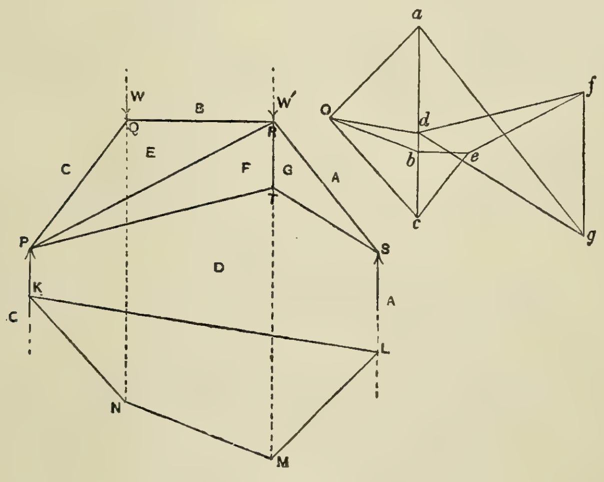

Fig. 361.

Draw ac equal to the sum of the two weights, and set off ab equal to the weight W. Assume any pole O, and join it to the points a, b, and c.

Construct a funicular polygon to determine the forces CD and DA, as follows:—

Draw KN parallel to Oc, NM parallel to Ob, and ML parallel to Oa. Join KL. Draw Od parallel to KL. Then cd = CD, and da = DA.

Draw lines from the points *a, d, b,* and *c* parallel to the several bars, to complete the force polygon, as described in the preceding problem.

The Centre of Gravity.—A body is built up of particles which are attracted towards the earth's centre by what are virtually parallel forces, called *gravity*.

The resultant of these parallel forces, the sum of which is the weight of the body, may be considered to be applied vertically at a point of the body called the *centre of gravity*. If this point be supported, the body will rest in whatever position it may be placed.

The centres of gravity of the following objects can be found graphically :—

Of a straight line.—Its middle point.

Of two particles of equal weight.—A point midway between them.

Of two particles of unequal weight.—Prob. 289.

Of three particles of unequal weight.—Prob. 292.

PLANE FIGURES.

Note.—Plane figures in this subject are generally called *laminæ*.

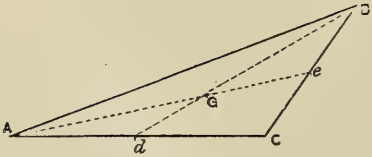

Fig. 362.

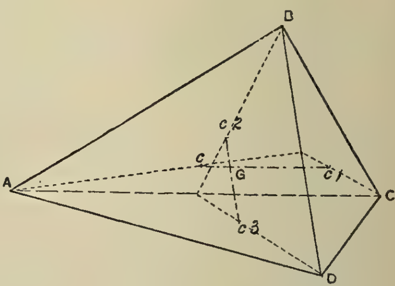

Fig. 363.

Of a triangle.—Fig. 362.

Bisect any two sides, and join the points to the opposite angles. The intersection of these two lines, as G, is the centre

of gravity. This point is always one-third of the length of the line joining the centre of the base to the apex: *e.g.* if we take AC as the base, *d*G is one-third of *d*B; or, if CB is taken for the base, then *e*G is one-third of *e*A.

Of parallelograms: a *square*, a *rectangle*, a *rhombus*, and a *rhomboid*.—The point of intersection of the two diagonals.

Of regular polygons.—The centre of the circumscribing circle.

Of any irregular quadrilateral figure.—Fig. 363.

Let ABCD be the figure. Join BD, and find the centres of gravity of the two triangles ABD and BCD, as *c* and *c*1 (Fig. 362), and join them. Join AC, and find the centres of gravity of the triangles ABC and ACD, as *c*2 and *c*3, and join them.

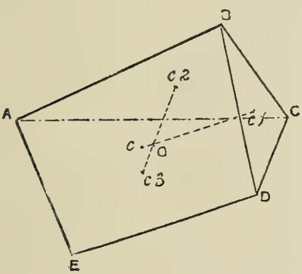

Fig. 364.

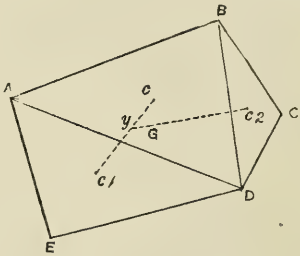

Fig. 365.

Where these lines intersect in G is the centre of gravity of the quadrilateral figure.

Of any irregular polygon.—There are two methods for obtaining the centres of gravity for these figures, as follows:—

1. Let ABCDE (Fig. 364) be the figure. Join BD. Find centres of gravity of the triangle BCD (Fig. 362) and the quadrilateral ABDE (Fig. 363), as *c* and *c*1, and join them. Join AC, and find *c*2 and *c*3 in the same manner, and join them. The intersection of these two lines gives G, the centre of gravity required.

2. Let ABCDE (Fig. 365) be the figure. Divide it into three triangles by joining AD and BD.

Find the centres of gravity of each triangle (Fig. 362), as *c* and *c*1, and join them. Let *c* and *c*1 be two particles repre-

senting the relative weights of their respective triangles, determined by their area. Their resultant acts through *y* (Prob. 289). Find the position of *c*2 (Fig. 362), and join it to *y*. Let *y* represent the weight of the quadrilateral ABDE, and *c*2 the weight of the triangle BCD. The resultant of these two forces (Prob. 289) acts through G. This is the centre of gravity of ABCDE.

If the polygon had more sides, additional triangles would be taken and treated in the same manner.

Of a semicircle.—Divide two-thirds of the square of the diameter of the circle by its circumference, and set off the result on a perpendicular from the centre of the base of the semicircle.

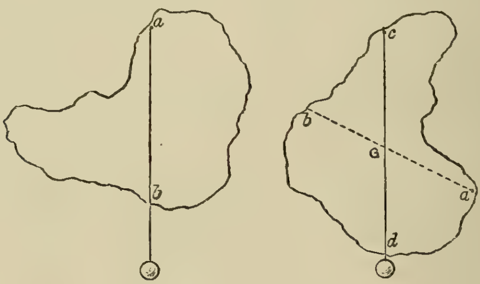

Fig. 366.

Of any irregular lamina.—Fig. 366. First suspend it freely from any point *a* of its circumference, and from the same point attach a plumb-line, and mark its position on the lamina, as *ab*. Then suspend it in like manner from any other point on its circumference, as *c*, and mark the position of the plumb-line, as *cd*. Where these lines intersect in G is the centre of gravity required.

SOLIDS.

Of a prism or cylinder.—Join the centres of gravity of the two ends by a line, and bisect it.

Of a pyramid or cone.—Join the centre of gravity of the base to its vertex by a line, and set off one-fourth of its length from the base.

DEPARTMENT OF SCIENCE AND ART

SYLLABUS

Science Subject I.—Practical Plane and Solid Geometry

The examination questions will be given with the object of testing the candidate's knowledge of the *principles* of the subject, and in order to obtain a class it will not be sufficient to have merely learned a few problems by heart. In awarding marks, the neatness and general style of the drawing will be taken into consideration.

In addition to the questions in Practical Geometry, a few questions in Graphic Arithmetic and Statics will be given in each stage. These questions will be alternative with and in addition to those in Practical Geometry, the candidate being allowed to choose some of them in place of questions in Plane or Solid Geometry. No marks will be given for Arithmetical or Algebraic solutions.

A candidate for examination in Geometrical Drawing must confine himself to Section I. A candidate for examination in the Elementary Stage of Practical Plane and Solid Geometry must take both Sections I. and II.

First Stage or Elementary Course

Section I.

Geometrical Drawing

	Reference to Problems in this Book. Plane Geometry.
1. Construction and use of plain scales and scales of chords.	Probs. 156-159, 165.
2. Proportional division of lines.	„ 9, 10, 11, and 118.
3. Mean third and fourth proportional to given lines.	„ 112-117.

	Reference to Problems in this Book.
	Plane Geometry.
4. Elementary constructions relating to lines and circles required in drawing out geometrical patterns and simple tracery.	Probs. 69 - 77 and Chap. 5.
5. Reduction and enlargement of plane figures.	Chap. 14.
6. Construction of regular polygons on a given side.	Probs. 40-43, 45 and 46.
7. Inscription of regular polygons in a given circle.	Probs. 37-39, 44, 47, and 48.
8. Construction of irregular polygons from given data.	Probs. 19 - 25 (see Exercises 5, 6, and 10, Chap. 3, p. 193).
9. Reduction of irregular figures to triangles and squares.	Probs. 124-130.
10. Elementary constructions relating to ellipses.	„ 178-182.
	Solid Geometry.
11. Plan, elevation, and section of cube, pyramid, prism, cylinder, cone, and sphere in simple positions.	Probs. 1-4, 29, 32, 35, and 39 ; and beginning of Chap. 30.

Section II.

Solid Geometry.

1. The principles of projection. Definitions of terms in general use, such as projector, plan, elevation, section, trace, etc.	Chap. 17, and up to Prob. 59, Chap. 23 ; also see note at bottom of p. 13.
2. Simple problems relating to lines.	Pp. 278-296.
3. Simple problems relating to planes.	Pp. 296-304.
4. Plan and elevation of simple solids resting on the horizontal plane.	Probs. 1-4, 29, 32, 35, and 39.
5. Plan and elevation of plane figures having given inclination of two sides or of plane and one side.	Probs. 122, 123 (1), 124 (1), and 125 (1).
6. Plan and elevation of simple solids having one edge in the horizontal plane, and an adjacent face inclined at a given angle ; or given the inclination of one face and one edge.	Probs. 5, 6, 13, 19, 20, and 31.
7. Sections of such solids by vertical and horizontal planes.	See commencement of Chap. 30.
In the elementary paper, given points or lines will be above the horizontal and in front of the vertical plane.	See commencement of Chap. 23.

	Reference to Problems in this Book. Solid Geometry.
1. The representation of numbers by lines.	Probs. 248-255.
2. The multiplication of numbers by construction.	,, 256-260.
3. The division of numbers by construction.	,, 261 and 262.
4. The determination of the square root of numbers by construction.	,, 267-270.

Second Stage or Advanced Course.

Plane Geometry.

	Plane Geometry.
1. The construction of plain and diagonal "scales" to different linear units, English and foreign.	Probs. 156-164.
2. The division of a line in given proportion, and in extreme and mean ratio. Construction for mean, third, and fourth proportionals, and for harmonic mean.	Probs. 11, 112-118, and 122.
3. The construction of polygons from adequate conditions of sides, angles, area, or perimeter.	Chap. 3 and p. 132.
4. The construction required in dealing with lines which intersect at points practically out of reach.	Probs. 31, 32, and 35.
5. Miscellaneous problems relating to lines, circles, and plane figures.	Chaps. 2, 4, 5, 6, 7, 8, 9, 13, and 14.
6. The delineation of plane curves, such as the ellipse, parabola, hyperbola, cycloid, spirals, etc.	Chap. 15, and Probs. 193, 201-203.
7. The construction of simple loci, both from geometrical and mechanical data, such as the practical setting out of curves described by particular parts of machines or link work.	Probs. 205, 206, and 208.
8. The construction of curves from observed or tabular data, such as curves of varying pressure, temperature, resistance, and so forth, involving the elementary geometrical use of abscissa and ordinate.	Prob. 209.

Solid Geometry.

	Solid Geometry.
1. Miscellaneous problems relating to lines and planes.	Chaps. 23 to 27.

	Reference to Problems in this Book. Solid Geometry.
2. Projections of the cube, prism, pyramid, tetrahedron, octohedron, having given— *a.* A plane connected with the solid, and a line lying in that plane. *b.* Two lines connected with the solid. *c.* Two planes ,, ,, ,, *d.* The heights of three points of the solid.	Chap. 18, and Probs. 17-22, 28, 77-79, and Chap. 29.
3. Sections of the above solids by vertical, horizontal, and inclined planes.	Chap. 30, up to Prob. 131.
4. Problems relating to the sphere, cone, and cylinder— *a.* Representation of those solids in given positions and in contact. *b.* Determination of tangent planes to them. *c.* Determination of their sections by planes under given conditions. *d.* Intersection of their surfaces when variously combined, or "interpenetration."	Chaps. 20, 21, Probs. 133-138, also Chap. 31, and Probs. 185, 188, 189, 229-235, 237-239.
5. The screw, delineation of the simple helix. The square and V-threaded screw (*Machine Drawing*).	Prob. 50.
6. Simple cases of cast shadows, the rays of light being supposed parallel.	Probs. 199-207.
7. Principles and practice of isometric projection.	Chap. 36.
N.B.—The solid geometry problems may be worked either by means of projections on two co-ordinate planes, or by *horizontal projection* with figured indices, which will often simplify the constructions. Either mode of representation will be employed for the data of the questions, as may be most convenient.	Chap. 32.

Graphic Arithmetic and Statics.

1. Multiplication and division of numbers by lines.	Probs. 256-262.
2. Determine by construction lines representing the values of such expressions, as— $\sqrt{m}$, $\sqrt{\frac{m}{n}}$, $\sqrt{\frac{1}{m}}$ etc. ; m and n being given numbers.	Probs. 267-270.
3. Representation of areas and volumes by lines.	Pp. 543-544.
4. Resolution of a given force in two directions.	Probs. 275-281.

	Reference to Problems in this Book.
	Solid Geometry.
5. Determination of the resultant of any number of forces acting at a point; or parallel.	Probs. 282-287, 289-295.
6. Graphic determination of the moment of a force, and of the resultant moment of several forces.	Probs. 288, 296-302.

Examination for Honours.

Plane Geometry.

	Plane Geometry.
1. Miscellaneous problems relating to lines and circles.	Chaps. 2, 5, 6, and 7.
2. Construction of plane figures from different data.	Chaps. 3, 4, 8, 9, 13, and 14.
3. Construction from different data of the ellipse, parabola, hyperbola, their tangents and centres of curvature.	Chap. 15.
4. Delineation of various curves which occur in the arts, or find place in the geometry of machines.	Pp. 60, 61, 78-80, 164, 176, 181, 184-188.
5. The cycloid, trochoid, epicycloid, hypocycloid, epitrochoid, hypotrochoid, with their tangents, normals, and centres of curvature.	Probs. 193-198.
6. The evolute and involute of the circle and ellipse.	Prob. 199.
7. The Archimedean and logarithmic spirals.	Pp. 177-181.
8. Interpretation of loci from geometrical and mechanical data.	Pp. 182-186, and Prob. 208.
9. Construction of curves obtained by observation, or given by self-recording instruments.	Prob. 209.

Solid Geometry.

	Solid Geometry.
1. Miscellaneous problems relating to lines and planes.	Chaps. 23-28, and pp. 402-419.
2. Use of contours in determining the intersection of surfaces.	Probs. 224, 228, 230-236.
3. Projection of solids in various positions.	Chaps. 18-22, 29, and Probs. 185, 188, 189.
4. Sections of solids and surfaces of various forms by planes.	Chap. 30.
5. Intersection of curved surfaces and interpenetration of solids.	Chap. 35.
6. Development of conical and cylindrical surfaces.	Probs. 50-53.
7. Representation of helical and twisted surfaces.	Probs. 50-53.

	Reference to Problems in this Book.
	Solid Geometry.
8. Tangent planes to *surfaces of the second degree.*	Probs. 145-148, 151, 154, 155, 182-184, 187.
9. Shadows cast by solids of various form on each other and on any plane, the rays of light being either parallel or convergent.	Pp. 444-469.
10. Principles of isometric projection. 11. Isometric projection of objects of various forms.	Chap. 36.
12. Principles of map projection.	Chap. 37.
13. Principles and practice of perspective (*Perspective*).	

Graphic Statics.

1. Representation of numbers, areas, and volumes by lines.	Chap. 38.
2. Composition and resolution of forces acting in any direction in a plane.	Probs. 275-282, 284-287, 293-295.
3. Graphic determination of centres of gravity of plane figures.	Pp. 572-574.
4. Determination of the resultant couple of a system of couples in one plane.	P. 566.
5. Construction of stress diagrams for simple braced structures variously loaded.	Pp. 566-572.

Appendix to Syllabus of Subject I. (Practical Geometry).

Teachers of Science Classes are strongly recommended to study the following brief outline of an elementary course of Descriptive Geometry, and to base their teaching upon it. The problems therein enumerated should be thoroughly mastered, as they illustrate important principles and involve constructions which are of constant occurrence. When a difficulty presents itself in realising the conditions of a problem, the student should be encouraged to extemporise a model by bending up stiff paper or card to represent the planes of projection, cutting out pieces of card or paper to represent planes in various positions, and using pencils, pieces of wire, or thread to represent lines. By this means the habit of thinking out a question will be developed.

	Solid Geometry.
1. Represent (by its plan and elevation) a point in space in all possible positions with respect to the planes of projection (*i.e.* above or below the horizontal plane, before or behind the vertical plane). Conversely,	Pp. 274-278.

	Reference to Problems in this Book.
	Plane Geometry.
given the projections of a point, state its position with respect to the planes of projection.	
2. Given the plan and elevation of a line, with its extremities situated in given positions with regard to the co-ordinate planes, determine— *a.* The traces. *b.* The true length, and inclination to each plane of projection. Conversely, draw the plan and elevation of a line of given length when inclined at a° to the horizontal, and b° to the vertical plane. *c.* Obtain a new elevation of the line on any assigned ground line.	Pp. 278-296, and Chap. 22.
3. Assume two intersecting lines and determine the real angle they contain.	Probs. 77, 78, and 167.
4. Draw the projection of a line which is parallel to a given line.	Probs. 79, 160.
5. Represent by their traces planes in the following positions :— *a.* At right angles to one plane of projection and inclined at a given angle to the other. *b.* At right angles to both planes of projection. *c.* Parallel to the ground line.	Pp. 296-299.
6. Given the traces of a plane in any position with respect to the planes of projection. Determine the inclination of the plane to both planes of projection and the real angle contained by the traces. Conversely, obtain the traces of a plane which makes given angles with the planes of projection.	Probs. 81-84.
7. Determine the intersection of two given planes and the real angle they contain. Conversely, determine the traces of a plane which makes a given angle with a given plane.	Chaps. 24, 25, also Probs. 166, 174, and 175.
8. Determine a plane parallel to a given plane and (*a*) containing a given point ; or (*b*) at a given distance from the given plane.	Probs. 94, 163 (2), and 164.
9. Determine the intersection of a given line and plane and the angle contained between them.	Chap. 26, and Prob. 176.

	Reference to Problems in this Book.
	Plane Geometry.
10. From a given point drop a perpendicular on a given plane. At a given point on a given plane erect a perpendicular of given length.	Probs. 103, 104, 168, and 169.
11. In a given plane place a horizontal line or a line having any given inclination. From a given point draw a line parallel to a given plane and inclined at a given angle.	Chap. 27, and Probs. 171-173.
12. Obtain the projections of any polygon, given the inclination of its plane and of a side, diagonal, or any line connected with it.	Chaps. 28 and 29.
13. Determine a plane (*a*) to contain three given points, (*b*) to contain two lines including a given angle and inclined at given angles. Obtain the projections of any polygon, given the inclination of two adjacent sides, of a diagonal and adjacent sides, or generally of any two intersecting lines connected with the polygon.	Probs. 115, 165, and 167; also Chaps. 27, 28, and 29.
14. Draw the plans of the cube, prism, pyramid, tetrahedron, octohedron (*a*) resting with one face on the horizontal plane, (*b*) one edge in the horizontal plane and an adjacent face inclined at a given angle. Make elevations of these solids on different ground lines and sections by different vertical planes.	Chap. 17, and pp. 209-213 and 223-227; also Chap. 22 and commencement of Chap. 30.
15. Given the plan and one elevation of any object of simple form, make a new elevation, or a section on any given line.	Chap. 22, and Prob. 126.

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